10. Quantum Mechanical Markov Processes

Quantum mechanics, since the very early times in the 1920's, has been recognised as a description of the world which contains an essentially statistical aspect. Hence, all quantum mechanics must be regarded as being some kind of stochastic process. However, what is essentially unique to quantum mechanics is the description in terms of *complex probability amplitudes*, the square of whose modulus gives the actual probability of occurrence of an event.

The formulation of a proper quantum mechanical probability theory, or of quantum mechanics in terms of appropriately defined stochastic processes in this generalised probability theory, is not the aim of this chapter. What is of interest is the introduction of the reader to the rather fascinating world which straddles the boundaries of quantum and classical probability theory. This world is the realm of quantum optics and quantum electronics, where there are statistical aspects arising from the intrinsic quantum nature of the system, as well as fluctuations arising from thermal effects. We shall show how the quantum mechanics of optical systems can be related closely to Markov jump processes in a suitably generalised form, which can themselves very frequently be related by means of what are known as *P-representations* or otherwise, as *phase-space methods*, to diffusion processes in the complex plane. These diffusion processes can describe quasiprobabilities which may be negative or complex, or they may define genuine positive probabilities. The situation is very similar to that of the Poisson representation of Sect. 7.7 which is itself, in fact, a restricted form of P-representation.

We will formulate this chapter as follows. We first outline the quantum mechanics of the harmonic oscillator and introduce the concept of coherent states, which are central to the task. We then define a quantum Markov process and show how generalised Master equations can be derived for these, in a manner similar to that of the adiabatic elimination methods of Chap. 6. From these generalised Master equations we can sometimes develop ordinary birth-death Master equations, and sometimes, by using P-representations, we can develop Fokker-Planck equations. Both methods allow us to apply all the apparatus of classical stochastic processes to these quantum mechanical systems.

10.1 Quantum Mechanics of the Harmonic Oscillator

We describe the Harmonic oscillator in terms of creation and destruction operators a^+ and a which satisfy the commutation relations

$$[a, a^+] \equiv aa^+ - a^+a = 1 \tag{10.1.1}$$

and from which it can be shown that there are eigenstates $|n\rangle$ of a^+a such that

$$a^+a|n\rangle = n|n\rangle \qquad n = 0,1,2,3,\dots \tag{10.1.2}$$

$$\left.\begin{aligned} a|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a^+|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned}\right\} \tag{10.1.3}$$

and

$$\langle n|m\rangle = \delta_{m,n}\,. \tag{10.1.4}$$

The operator N is defined by

$$N = a^+a \tag{10.1.5}$$

and is known as the number operator since from (10.1.2), its eigenvalues are the integers n.

The Harmonic oscillator itself is defined by the Hamiltonian

$$H = (a^+a + \tfrac{1}{2})\hbar\omega \qquad \text{where} \tag{10.1.6}$$

$$h = 2\pi\hbar \tag{10.1.7}$$

is Planck's constant and ω is a frequency. The eigenstates of H are $|n\rangle$ of course, and the eigenvalues of H are

$$E_n = (n + \tfrac{1}{2})\hbar\omega\,. \tag{10.1.8}$$

Dynamics is introduced by Schrödinger's equation which determines the time development of any physical state $|A, t\rangle$.

It takes the well known form

$$H|A, t\rangle = \mathrm{i}\hbar\partial_t|A, t\rangle\,. \tag{10.1.9}$$

The orthonormality property (10.1.4) means that we can expand any state in terms of the energy eigenstates $|n\rangle$,

$$|A, t\rangle = \sum_n |n\rangle\,\langle n|A, t\rangle \tag{10.1.10}$$

and hence,

$$\left.\begin{aligned} \mathrm{i}\hbar\partial_t|A, t\rangle &= \mathrm{i}\hbar\sum_n |n\rangle\partial_t\langle n|A, t\rangle \\ &= \sum_n H|n\rangle\,\langle n|A, t\rangle \\ &= \sum_n (n + \tfrac{1}{2})\hbar\omega|n\rangle\,\langle n|A, t\rangle \end{aligned}\right\} \tag{10.1.11}$$

so that

$$\begin{aligned}\langle n | A, t\rangle &= \mathrm{e}^{-\mathrm{i}E_n t/\hbar}\langle n | A, 0\rangle \\ &= \mathrm{e}^{-\mathrm{i}(n+1/2)\omega t}\langle n | A, 0\rangle\,.\end{aligned} \tag{10.1.12}$$

Hence, the time development of an arbitrary state is now completely determined.

10.1.1 Interaction with an External Field

A semiclassical problem is that of the interaction of the harmonic oscillator with an external field. Without going into the physics too deeply, we simply state that this is done by modifying the Hamiltonian (10.1.6) to

$$H(\alpha) = [(a^+a + \tfrac{1}{2}) - (\alpha a^+ + \alpha^* a) + |\alpha|^2]\hbar\omega \tag{10.1.13}$$

where α is a complex number. The three parts may be regarded, respectively, as the harmonic oscillator energy, the interaction energy between the driving field α and the oscillator, and finally the (constant) energy of the driving field.

$H(\alpha)$ can be advantageously rewritten:

$$H(\alpha) = \hbar\omega[(a-\alpha)^+(a-\alpha) + \tfrac{1}{2}]\,. \tag{10.1.14}$$

It is obvious that the operators $(a-\alpha)^+$ and $(a-\alpha)$ obey the same commutation relation as a^+ and a, since α is a mere complex constant. Hence, the energy eigenstates have the same form, since the existence of states $|n\rangle$ follows from the commutation relation only.

Equation (10.1.3) can be used to define the ground state or vacuum state, $|0\rangle$ of a^+a by

$$a|0\rangle = 0\,. \tag{10.1.15}$$

The corresponding equation for the shifted operators $(a-\alpha)^+$, $(a-\alpha)$ is

$$a|\alpha\rangle = \alpha|\alpha\rangle\,. \tag{10.1.16}$$

We can check from (10.1.3) that a solution for $|\alpha\rangle$ is

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{10.1.17}$$

where the precise factor $\exp(-\tfrac{1}{2}|\alpha|^2)$ is chosen so that

$$\langle\alpha|\alpha\rangle = 1\,. \tag{10.1.18}$$

The states $|\alpha\rangle$ were devised by *Glauber* [10.1], and are known as *coherent states.* When the harmonic oscillator is regarded as a model of the radiation field in a single-mode system, the coherent state can be regarded as a quantum mechanical state which approaches a classical state.

The energy eigenstates of the Hamiltonian $H(\alpha)$ will now be $|n;\alpha\rangle$ and will have the same properties as (10.1.2–4) but are written in terms of the shifted operators $(a-\alpha)^+$ and $(a-\alpha)$.

10.1.2 Properties of Coherent States

We list the most important properties; no proof is given if the result is a simple matter of substitution of definitions.

a) Definition: $|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,.$ (10.1.19)

b) Scalar Product: $\langle\alpha|\beta\rangle = \exp(\alpha^*\beta - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2)$ (10.1.20)

$$|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha-\beta|^2)\,. \tag{10.1.21}$$

c) Completeness Formula

$$1 = \frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle\,\langle\alpha| \tag{10.1.22}$$

Here,

$$\begin{aligned} \alpha &= \alpha_x + \mathrm{i}\alpha_y\,, \\ d^2\alpha &= d\alpha_x d\alpha_y \end{aligned} \tag{10.1.23}$$

and the integral is over the whole complex plane. We prove this. For, if $|A\rangle$ is an arbitrary vector, then write

$$|A\rangle = \sum_n A_n|n\rangle \tag{10.1.24}$$

so that

$$\frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle\,\langle\alpha|A\rangle = \frac{1}{\pi}\sum_n\int A_n|\alpha\rangle\,\langle\alpha|n\rangle d^2\alpha\,. \tag{10.1.25}$$

Substitute (10.1.19) and change to polar coordinates by

$$\alpha = r\mathrm{e}^{\mathrm{i}\theta} \tag{10.1.26}$$

$$d^2\alpha = r\,dr\,d\theta\,. \tag{10.1.27}$$

Hence,

$$\begin{aligned} (10.1.25) &= \frac{1}{\pi}\sum_{m,n}\int A_n \mathrm{e}^{-r^2} r^{n+m}\,\mathrm{e}^{\mathrm{i}(m-n)\theta}(n!\,m!)^{-1/2}|m\rangle r\,dr\,d\theta \\ &= 2\sum_n\int A_n\mathrm{e}^{-r^2}r^{2n+1}(n!)^{-1}|n\rangle\,dr \end{aligned} \tag{10.1.28}$$

where we used

$$2\delta_{n,m} = \frac{1}{\pi}\int d\theta\, e^{i(m-n)\theta}\,. \tag{10.1.29}$$

Now noting

$$\int_0^\infty dr\, e^{-r^2} r^{2n+1} = n!/2\,, \tag{10.1.30}$$

we find

$$(10.1.25) = \sum_n A_n|n\rangle = |A\rangle\,. \tag{10.1.31}$$

Formulae (10.1.21,22) together indicate that *the coherent states are not orthogonal* for different α and β, and that since there is a factor $1/\pi$ in front of the integral (10.1.22) *the coherent states are overcomplete* [in fact, for any r, (10.1.19) shows that we can write

$$|n\rangle = \exp(\tfrac{1}{2}r^2)r^{-n}\sqrt{n!}\int d\theta\, e^{-in\theta}|\alpha\rangle \tag{10.1.32}$$

which indicates that the states for any fixed $r = |\alpha|$ are complete]. This overcompleteness is, however, not a disadvantage because of the very simple connection between coherent states and the physics of classical fields, and because of the fact that the *Bargmann states*, defined by

$$\|\alpha\rangle = \exp(\tfrac{1}{2}|\alpha|^2)|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,, \tag{10.1.33}$$

are analytic functions of α. This property is very important in what follows.

d) Expansion of Arbitrary States in Terms of Coherent States

Consider an arbitrary state $|f\rangle$. Then using the completeness relation, (10.1.22)

$$|f\rangle = \frac{1}{\pi}\int d^2\alpha|\alpha\rangle f(\alpha^*)\exp(-\tfrac{1}{2}|\alpha|^2)\,, \tag{10.1.34}$$

where

$$f(\alpha^*) = \langle\alpha|f\rangle\exp(\tfrac{1}{2}|\alpha|^2) = \langle\alpha\|f\rangle \tag{10.1.35}$$

is *an analytic function of α^**. With this proviso, the expansion (10.1.34) is unique. If functions of both α^* and α are permitted, the expansion is no longer unique, as Glauber shows.

The scalar product of two states $|f\rangle$ and $|g\rangle$ is straightforwardly shown to be

$$\langle g|f\rangle = \frac{1}{\pi}\int d^2\alpha\,[g(\alpha^*)]^* f(\alpha^*)\,e^{-|\alpha|^2} \tag{10.1.36}$$

which is obviously a Hilbert space of analytic functions. It provides, in fact, the soundest mathematical starting point for the study of the harmonic oscillator, the creation destruction operators and all the formalism of this chapter.

e) Expansion of an Operator in Coherent States

Consider any operator T in the quantum Hilbert space. Using the identity resolution twice,

$$\begin{aligned} T = 1\cdot T\cdot 1 &= \frac{1}{\pi^2}\int d^2\alpha d^2\beta\, |\alpha\rangle\langle\alpha|T|\beta\rangle\langle\beta| \\ &= \frac{1}{\pi^2}\int d^2\alpha d^2\beta |\alpha\rangle\langle\beta| T(\alpha^*, \beta)\exp(-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2) \end{aligned} \tag{10.1.37}$$

where

$$\begin{aligned} T(\alpha^*, \beta) &= \exp(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2)\langle\alpha|T|\beta\rangle \\ &= \langle\alpha\|T\|\beta\rangle \end{aligned} \tag{10.1.38}$$

and from the analyticity of the states $\|\alpha\rangle$, $\|\beta\rangle$, we see that *$T(\alpha^*, \beta)$ is an analytic function of α^* and β, and, with this proviso, is unique.* Notice, for example, that if

$$T = (a^+)^m(a^n)\,, \tag{10.1.39}$$

then

$$\left.\begin{aligned} T(\alpha^*, \beta) &= \langle\alpha|(a^+)^m(a^n)|\beta\rangle\exp(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2) \\ &= (\alpha^*)^m(\beta)^n\langle\alpha|\beta\rangle\exp(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2) \\ &= (\alpha^*)^m(\beta)^n\exp(\alpha^*\beta)\,. \end{aligned}\right\} \tag{10.1.40}$$

f) Any Operator *T* is Determined by Its Expectation in all Coherent States

For

$$\langle\alpha|T|\alpha\rangle = \sum_{n,m}\langle n|T|m\rangle\, e^{|\alpha|^2}(\alpha^*)^n(\alpha)^m/\sqrt{n!m!}$$

so that

$$\langle n|T|m\rangle = \sqrt{n!m!}\,\frac{\partial^n}{\partial\alpha^{*n}}\frac{\partial^m}{\partial\alpha^m}(e^{\alpha\alpha^*}\langle\alpha|T|\alpha\rangle)\,. \tag{10.1.41}$$

The derivatives are formal derivatives and, as in analytic function theory, are to be interpreted as

$$\left.\begin{aligned} &\alpha = x + iy \\ &\frac{\partial}{\partial\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ &\frac{\partial}{\partial\alpha^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{aligned}\right\} \tag{10.1.42}$$

in which case, $\langle n|T|m\rangle$ are the combinations of coefficients of power series in real variables x and y.

g) Coherent States are Eigenstates of a
Namely,

$$\left.\begin{aligned} a|\alpha\rangle &= \alpha|\alpha\rangle \\ &\text{and} \\ \langle\alpha|a^+ &= \langle\alpha|\alpha^* \end{aligned}\right\} \tag{10.1.43}$$

which are proved directly from the definition and was the original basis for investigating them. In evaluating matrix elements, *normal products* of operators in which all destruction operators stand to the right of creation operators, are useful. Thus,

$$\left.\begin{aligned} \langle\alpha|a^+aa^+|\beta\rangle &= \langle\alpha|a^+a^+a + a^+[a, a^+]|\beta\rangle \\ &= \langle\alpha|a^+a^+a + a^+|\beta\rangle \\ &= (\alpha^{*2}\beta + \alpha^*)\langle\alpha|\beta\rangle\,. \end{aligned}\right\} \tag{10.1.44}$$

The symbol: : around an expression means that it is to be considered a normal product: thus,

$$:(a + a^+)\,(a + a^+): = a^{+2} + a^2 + 2a^+a\,. \tag{10.1.45}$$

h) Poissonian Number Distribution of Coherent States
The state $|n\rangle$ is known as an "*n quantum state*" since its energy is $n\hbar\omega$ more than that of the vacuum $|0\rangle$, the zero quantum state. In quantum mechanics, therefore, the probability of observing n quanta in a coherent state $|\alpha\rangle$ is

$$P_\alpha(n) = |\langle n|\alpha\rangle|^2 = \left|\exp\left(-\tfrac{1}{2}\alpha^2\right)\frac{\alpha^n}{\sqrt{n!}}\right|^2 \tag{10.1.46}$$

$$= \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!} \tag{10.1.47}$$

which is a Poisson distribution with mean $|\alpha|^2$.

Since the number n corresponds to the eigenvalue of the number operator N, we have

$$\langle N\rangle = \langle\alpha|N|\alpha\rangle = \sum_n nP(n) = |\alpha|^2 \tag{10.1.48}$$

$$\begin{aligned} \langle N^2\rangle &= \langle\alpha|a^+aa^+a|\alpha\rangle \\ &= \langle\alpha|a^+a^+aa + a^+[a, a^+]a|\alpha\rangle \\ &= |\alpha|^4 + |\alpha|^2\,. \end{aligned}$$

Hence,

$$\langle N(N-1)\rangle = |\alpha|^4 = \langle N\rangle^2 \tag{10.1.49}$$

as required for a Poisson. The Poissonian nature of the probability distribution of quanta in a coherent state is what provides the link with the Poisson representation.

10.2 Density Matrix and Probabilities

The usual quantum mechanical formula, that the mean of a quantity M in a state $|\psi\rangle$ is given by

$$\langle M\rangle = \langle\psi|M|\psi\rangle\,, \tag{10.2.1}$$

provides only for experiments in which the identical quantum state is measured repeatedly. A more usual possibility is that, due to the random nature of state preparation, which may arise from thermal effects or simply from inadequate preparation apparatus, we measure means in different states $|\psi_a\rangle$ each time we do a measurement. Then in this case, if each state occurs with probability $P(a)$, this probability not arising from quantum effects but simply arising from the randomness of state preparation, the measured mean is

$$\langle M\rangle = \sum P(a)\langle\psi_a|M|\psi_a\rangle\,. \tag{10.2.2}$$

One now introduces the density matrix (or operator) ρ by

$$\rho = \sum_a P(a)|\psi_a\rangle\langle\psi_a| \tag{10.2.3}$$

through which

$$\langle M\rangle = \mathrm{Tr}\{\rho M\}\,. \tag{10.2.4}$$

Here, for any operator B, we define the trace operation Tr by

$$\mathrm{Tr}\{B\} = \sum_n \langle n|B|n\rangle \tag{10.2.5}$$

so that

$$\begin{aligned}\mathrm{Tr}\{\rho M\} &= \sum_{n,a} P(a)\langle n|\psi_a\rangle\langle\psi_a|M|n\rangle\\ &= \sum_{n,a} P(a)\langle\psi_a|M|n\rangle\,\langle n|\psi_a\rangle\\ &= \sum_a P(a)\langle\psi_a|M|\psi_a\rangle\\ &= \langle M\rangle\,.\end{aligned}$$

An important property of the trace is its invariance under cyclic permutations of factors so that

$$\mathrm{Tr}\{\rho M\} = \mathrm{Tr}\{M\rho\} \tag{10.2.6}$$

$$\mathrm{Tr}\{ABCD\} = \mathrm{Tr}\{BCDA\}\,, \text{ etc.} \tag{10.2.7}$$

Important Density Matrix Properties:

i) $\mathrm{Tr}\{\rho\} = 1$ (10.2.8)

for

$$\mathrm{Tr}\{\rho\} = \sum_a P_a\langle\psi_a|\psi_a\rangle = \sum_a P_a = 1\,. \tag{10.2.9}$$

ii) ρ is positive semidefinite; for any state $|A\rangle$,

$$\langle A|\rho|A\rangle = \sum P_a|\langle A|\psi_a\rangle|^2 \geqslant 0\,. \tag{10.2.10}$$

iii) If ρ corresponds to a pure state, then $\rho^2 = \rho$ and conversely, for a pure state, $P_a = \delta_{a,a_0}$ for some a_0, so that

$$\rho^2 = |\psi_{a_0}\rangle\langle\psi_{a_0}|\psi_{a_0}\rangle\langle\psi_{a_0}| = \rho\,. \tag{10.2.11}$$

Conversely, consider

$$\rho^2 = \sum P_a P_b|\psi_a\rangle\langle\psi_b|\langle\psi_a|\psi_b\rangle\,. \tag{10.2.12}$$

This equals ρ only if

$$P_a P_b\langle\psi_a|\psi_b\rangle = 0 \qquad \text{for } a \neq b$$
$$P_a^2\langle\psi_a|\psi_a\rangle = P_a$$

But

$$\langle\psi_a|\psi_a\rangle = 1\,,$$

hence,

$$P_a^2 = P_a \Longrightarrow P_a = 1 \quad \text{or} \quad 0\,.$$

But since $\sum P_a = 1$, only one P_a can be one, and the others zero, which corresponds to a pure state.

iv) $\mathrm{Tr}\{\rho^2\} \leqslant \mathrm{Tr}\{\rho\}$, with equality only for a pure state.

For,

$$\mathrm{Tr}\{\rho^2\} = \sum_{a,b} P_a P_b\,|\langle\psi_a|\psi_b\rangle|^2 \tag{10.2.13}$$

and since

$$|\langle\psi_a|\psi_b\rangle|^2 \leqslant 1 \tag{10.2.14}$$

$$\sum_b P_b |\langle\psi_a|\psi_b\rangle|^2 \leqslant 1, \quad \text{for any } a\,. \tag{10.2.15}$$

Therefore,

$$\mathrm{Tr}\{\rho^2\} \leqslant \sum_a P_a = \mathrm{Tr}\{\rho\} \tag{10.2.16}$$

and it is clear that (10.2.15) can only be an equality if

$$P_b = \delta_{b,a_0}$$

for some a_0, i.e., ρ is a pure state.

10.2.1 Von Neumann's Equation

The Schrödinger equation is

$$H|\psi\rangle = \mathrm{i}\hbar\partial_t|\psi\rangle \tag{10.2.17}$$

for any state. A corresponding equation for the density operator can be derived. For,

$$\begin{aligned}\partial_t\rho &= \sum_a P_a[(\partial_t|\psi_a\rangle)\langle\psi_a| + |\psi_a\rangle(\partial_t\langle\psi_a|)]\\ &= \frac{1}{\mathrm{i}\hbar}(H\rho - \rho H)\,.\end{aligned}$$

That is,

$$\boxed{[H, \rho] = \mathrm{i}\hbar\partial_t\rho} \tag{10.2.18}$$

which is *von Neumann's equation* or the *quantum Liouville equation.* Von Neumann's equation can be exponentiated to give the formal solution

$$\rho(t) = \exp(-\mathrm{i}Ht/\hbar)\rho(0)\exp(\mathrm{i}Ht/\hbar)\,. \tag{10.2.19}$$

10.2.2 Glauber-Sudarshan P-Representation

Glauber [10.1] and *Sudarshan* [10.2] introduced a representation for the density matrix now known as the *Glauber-Sudarshan P-representation.* One assumes that ρ can be written as

$$\rho = \int d^2\alpha\, P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha| \tag{10.2.20}$$

where $P(\alpha, \alpha^*)$ will play the role of a quasiprobability. Questions of existence of this P-representation we leave aside for the moment.

Notice that since

$$1 = \langle\alpha|\alpha\rangle = \sum_n \langle\alpha|n\rangle\langle n|\alpha\rangle\,,$$

$$\mathrm{Tr}\{\rho\} = 1 = \int d^2\alpha\, P(\alpha,\alpha^*) \sum_n \langle n|\alpha\rangle\langle\alpha|n\rangle$$

i.e.,

$$\boxed{1 = \int d^2\alpha\, P(\alpha,\alpha^*)\,.} \tag{10.2.21}$$

Further, for any normal product $(a^+)^r a^s$,

$$\begin{aligned}\mathrm{Tr}\{(a^+)^r a^s \rho\} &= \mathrm{Tr}\{a^s \rho (a^+)^r\}\\ &= \int d^2\alpha\, P(\alpha,\alpha^*) \sum_n \langle n|a^s|\alpha\rangle\,\langle\alpha|(a^+)^r|n\rangle\,,\end{aligned}$$

i.e.,

$$\boxed{\langle (a^+)^r a^s\rangle = \int d^2\alpha (\alpha^*)^r \alpha^s P(\alpha,\alpha^*)\,.} \tag{10.2.22}$$

Thus, the quantity $P(\alpha, \alpha^*)$ plays the role of a kind of probability density for the variables α and α^*, in that the means of normally ordered products of quantum operators are simple moments of $P(\alpha, \alpha^*)$.

The conditions under which a Glauber-Sudarshan P-representation exists are problematical. *Klauder* and *Sudarshan* [10.3] have shown that, providing sufficiently singular generalised functions are chosen, it always exists. It certainly does not always exist as a positive function, nor indeed as a smooth function.

For the moment we will leave aside these questions, the answers to which are very similar to those for the various Poisson representations of Chap. 7.

10.2.3 Operator Correspondences

We know that

$$\left.\begin{aligned}&a|\alpha\rangle = \alpha|\alpha\rangle\\ &\text{and}\\ &\langle\alpha|a^+ = \alpha^*\langle\alpha|\,.\end{aligned}\right\} \tag{10.2.23}$$

For the other possible ways of acting with the a, a^+, it is convenient to use the Bargmann states $\|\alpha\rangle$ so that

$$\begin{aligned}a^+\|\alpha\rangle &= \sum_n \frac{\alpha^n}{\sqrt{n!}}\sqrt{n+1}\;|n+1\rangle\\ &= \frac{\partial}{\partial\alpha}\|\alpha\rangle\,.\end{aligned} \tag{10.2.24}$$

Similarly,

$$\langle\alpha\|a = \frac{\partial}{\partial\alpha^*}\langle\alpha\| \, .$$

Hence, we can write for a ρ representable by a P-representation which can be conveniently transformed, for this purpose, to the form

$$\rho = \int d^2\alpha \|\alpha\rangle\langle\alpha\| e^{-\alpha\alpha^*} P(\alpha, \alpha^*) \, ,$$

a simple equation,

$$a^+\rho = \int d^2\alpha \frac{\partial}{\partial\alpha}(\|\alpha\rangle)\,\langle\alpha\| e^{-\alpha\alpha^*}\, P(\alpha, \alpha^*) \, ,$$

and integrating by parts

$$= \int d^2\alpha \|\alpha\rangle\,\langle\alpha\|\, e^{-\alpha\alpha^*}\left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha, \alpha^*) \, . \tag{10.2.25}$$

We thus can make an *operator correspondence* between a^+ and $\alpha^* - \partial/\partial\alpha$. A similar formula holds for a. Summarising, with the obvious correspondences arising from (10.2.23,24), we have

$$\boxed{\begin{aligned} a\rho &\leftrightarrow \alpha P(\alpha, \alpha^*) \\ a^+\rho &\leftrightarrow \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha, \alpha^*) \\ \rho a &\leftrightarrow \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P(\alpha, \alpha^*) \\ \rho a^+ &\leftrightarrow \alpha^* P(\alpha, \alpha^*) \, . \end{aligned}} \tag{10.2.26}$$

10.2.4 Application to the Driven Harmonic Oscillator

We consider the Hamiltonian

$$H = \hbar\omega(a^+a + \tfrac{1}{2}) + (\lambda a^+ + \lambda^* a) \tag{10.2.27}$$

for which the quantum Liouville equation is

$$i\hbar\frac{\partial\rho}{\partial t} = \hbar\omega[a^+a, \rho] + \lambda[a^+, \rho] + \lambda^*[a, \rho] \, . \tag{10.2.28}$$

We now turn this into an equation for $P(\alpha, \alpha^*)$ by using the operator correspondences (10.2.26). Thus,

$$a^+a\rho \rightarrow \left(\alpha^* - \frac{\partial}{\partial\alpha}\right)\alpha P \tag{10.2.29}$$

$$-\rho a^{+}a \rightarrow -\left(\alpha - \frac{\partial}{\partial\alpha^{*}}\right)\alpha^{*}P \tag{10.2.30}$$

[notice that the order of the operators in (10.2.30) reverses, since acting on ρ they operate from the right, whereas on P, they operate from the left]:

$$\begin{aligned}[a^{+}, \rho] &\rightarrow \left[\left(\alpha^{*} - \frac{\partial}{\partial\alpha}\right) - \alpha^{*}\right]P \\ &= -\frac{\partial}{\partial\alpha}P\,.\end{aligned} \tag{10.2.31}$$

Similarly,

$$[a, \rho] \rightarrow \frac{\partial}{\partial\alpha^{*}}P \tag{10.2.32}$$

so that we find

$$\frac{\partial P}{\partial t} = \mathrm{i}\left(-\omega\frac{\partial}{\partial\alpha}\alpha + \omega\frac{\partial}{\partial\alpha^{*}}\alpha^{*} - \frac{\lambda}{\hbar}\frac{\partial}{\partial\alpha} + \frac{\lambda^{*}}{\hbar}\frac{\partial}{\partial\alpha^{*}}\right)P\,. \tag{10.2.33}$$

This corresponds to a Liouville equation for the variables. A word of caution. It is tempting to treat α and α^{*} as independent variables which is not strictly true, and in writing all the above correspondences, one should really write

$$\left.\begin{aligned}\alpha &= x + \mathrm{i}y \\ \alpha^{*} &= x - \mathrm{i}y \\ \frac{\partial}{\partial\alpha} &= \tfrac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial\alpha^{*}} &= \tfrac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)\end{aligned}\right\} \tag{10.2.34}$$

and

$$\lambda = \mu + \mathrm{i}\nu\,,$$

and in terms of these real variables, (10.2.33) becomes

$$\frac{\partial P}{\partial t} = \left[\frac{\partial}{\partial x}(\omega y + \nu/\hbar) - \frac{\partial}{\partial y}(\omega x + \mu/\hbar)\right]P \tag{10.2.35}$$

which is a Liouville equation, equivalent to the differential equations

$$\begin{aligned}\frac{dx}{dt} &= -\,\omega y - \nu/\hbar \\ \frac{dy}{dt} &= \omega x + \mu/\hbar\end{aligned} \tag{10.2.36}$$

which are equivalent to

$$\frac{d\alpha}{dt} = \mathrm{i}(\omega\alpha + \lambda/\hbar) \tag{10.2.37}$$

with the solution

$$\alpha = -\lambda/\hbar\omega + \xi \mathrm{e}^{\mathrm{i}\omega t}. \tag{10.2.38}$$

The solution for P is, assuming a deterministic initial condition,

$$P(\alpha, \alpha^*, t) = \delta\left(x - \mathrm{Re}\left\{-\frac{\lambda}{\hbar\omega} + \xi\mathrm{e}^{\mathrm{i}\omega t}\right\}\right) \delta\left(y - \mathrm{Im}\left\{-\frac{\lambda}{\hbar\omega} + \xi\mathrm{e}^{\mathrm{i}\omega t}\right\}\right) \tag{10.2.39}$$

$$= \delta^2(\alpha + \lambda/\hbar\omega - \xi\mathrm{e}^{\mathrm{i}\omega t}), \tag{10.2.40}$$

where the complex delta function is understood to mean (10.2.39).

Notice that λ may depend on time, in which case (10.2.37) becomes

$$\frac{d\alpha}{dt} = \mathrm{i}[\omega\alpha + \lambda(t)/\hbar] \tag{10.2.41}$$

whose solution is

$$\alpha(t) = \alpha(0)\,\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{i}\int_0^t dt'\, \mathrm{e}^{\mathrm{i}\omega(t-t')}\lambda(t')/\hbar \tag{10.2.42}$$

and the corresponding P is

$$P(\alpha, \alpha^*, t) = \delta^2[\alpha - \alpha(t)]. \tag{10.2.43}$$

10.2.5 Quantum Characteristic Function

The Fourier transform of $P(\alpha, \alpha^*)$ would provide a natural characteristic function like that introduced in Sect. 2.6. Let us define

$$\chi(\lambda, \lambda^*) = \int d^2\alpha \exp(\lambda\alpha^* - \lambda^*\alpha)\, P(\alpha, \alpha^*). \tag{10.2.44}$$

Notice that if

$$\alpha = x + \mathrm{i}y$$
$$\lambda = \mu + \mathrm{i}\nu,$$

then

$$\lambda\alpha^* - \lambda^*\alpha = 2\mathrm{i}(\nu x - y\mu) \tag{10.2.45}$$

so that (10.2.44) is a Fourier transform in two real variables. It is also possible to write

$$\boxed{\chi(\lambda, \lambda^*) = \mathrm{Tr}\{\rho \exp(\lambda a^+) \exp(-\lambda^* a)\}} \tag{10.2.46}$$

which can serve as a general definition of the quantum characteristic function. Notice that for any operator A,

$$\begin{aligned}\mathrm{Tr}\{A\} &= \sum_n \langle n|A|n\rangle = \frac{1}{\pi}\int d^2\alpha \sum_n \langle n|\alpha\rangle \langle\alpha|A|n\rangle \\ &= \frac{1}{\pi}\int d^2\alpha \sum_n \langle\alpha|A|n\rangle \langle n|\alpha\rangle ,\end{aligned}$$

i.e.,

$$\boxed{\mathrm{Tr}\{A\} = \frac{1}{\pi}\int d^2\alpha\, \langle\alpha|A|\alpha\rangle .} \tag{10.2.47}$$

We now introduce the *Baker-Hausdorff* formula [10.4]. For any two operators A and B such that their commutator $[A, B]$ commutes with both of them, One can write

$$\boxed{\begin{aligned}\exp(A + B) &= \exp(A)\exp(B)\exp(-\tfrac{1}{2}[A, B]) \\ &= \exp(B)\exp(A)\exp(\tfrac{1}{2}[A, B]) .\end{aligned}} \tag{10.2.48}$$

Noting

$$[\lambda a^+, -\lambda^* a] = |\lambda|^2 ,$$

we see

$$\exp(\lambda a^+)\exp(-\lambda^* a) = \exp(-\lambda^* a)\exp(\lambda a^+)\exp(|\lambda|^2) .$$

Hence,

$$\begin{aligned}\mathrm{Tr}\{\rho \exp(\lambda a^+)\exp(-\lambda^* a)\} &= \exp(|\lambda|^2)\mathrm{Tr}\{\rho\exp(-\lambda^* a)\exp(\lambda a^+)\} \\ &= \exp(|\lambda|^2)\mathrm{Tr}\{\exp(\lambda a^+)\,\rho\exp(-\lambda^* a)\} \\ &= \frac{\exp(|\lambda|^2)}{\pi}\int d^2\alpha\langle\alpha|\exp(\lambda a^+)\,\rho\exp(-\lambda^* a)|\alpha\rangle\end{aligned}$$

so

$$\boxed{\chi(\lambda, \lambda^*) = \frac{\exp(|\lambda|^2)}{\pi}\int d^2\alpha \exp(\lambda\alpha^* - \lambda^*\alpha)\langle\alpha|\rho|\alpha\rangle .} \tag{10.2.49}$$

Since

$$\langle\alpha|\rho|\alpha\rangle > 0 \tag{10.2.50}$$

and

$$\mathrm{Tr}\{\rho\} = \frac{1}{\pi}\int d^2\alpha \, \langle\alpha|\rho|\alpha\rangle = 1\,, \tag{10.2.51}$$

(10.2.49) is a Fourier transform of a function $\langle\alpha|\rho|\alpha\rangle$ which satisfies the conditions of a probability function. Hence, the Fourier transform $\chi(\lambda,\lambda^*)\exp(-|\lambda|^2)$ is its corresponding characteristic function and from Sect. 2.6, it determines $\langle\alpha|\rho|\alpha\rangle$ with probability one.

From Sect. 10.1.2, we then see that $\langle\alpha|\rho|\alpha\rangle$ determines ρ. Hence, $\chi(\lambda, \lambda^*)$ determines ρ with probability one.

10.3 Quantum Markov Processes

We now briefly develop a simplified form of the quantum theory of damping. This requires some knowledge of quantum statistics.

10.3.1 Heat Bath

Damping, in practice, occurs because a system interacts with another very large system known as a *Heat Bath*, into which the energy of the system is dissipated. However, noise arises also, since the heat bath distributes some of its energy back into the system.

As a model of a heat bath, consider a large number of independent harmonic oscillators with operators Γ_i and Hamiltonian

$$H_\mathrm{B} = \sum_i \hbar\omega_i(\Gamma_i^+\Gamma_i + \tfrac{1}{2})\,. \tag{10.3.1}$$

This system does, in fact, possess a stationary density matrix; indeed, any positive function of H_B is satisfactory. Statistical mechanics allows us to choose a *canonical ensemble* in which the density operator is

$$\rho(T) = \exp(-H_\mathrm{B}/kT)/\mathrm{Tr}\{\exp(-H_\mathrm{B}/kT)\}\,, \tag{10.3.2}$$

where T is the temperature of the heat bath. It is trivial that

$$[H_\mathrm{B}, \rho(T)] = \mathrm{i}\hbar\partial_t\rho = 0\,. \tag{10.3.3}$$

Because H_B is the sum of terms which commute with each other, one can write

$$\rho = \prod_i \rho_i(T)$$

with

$$\rho_i(T) = \exp\left[-\frac{\hbar\omega_i}{kT}(\Gamma_i^+\Gamma_i + \tfrac{1}{2})\right]\Big/\mathrm{Tr}\left\{\exp\left[-\frac{\hbar\omega_i}{kT}(\Gamma_i^+\Gamma_i + \tfrac{1}{2})\right]\right\}\,. \tag{10.3.4}$$

Further define

$$Z_i = \mathrm{Tr}\left\{\exp\left[-\frac{\hbar\omega_i}{kT}(\Gamma_i^+\Gamma_i + \tfrac{1}{2})\right]\right\} = \sum_n \exp\left\{-\frac{(n+\frac{1}{2})\hbar\omega_i}{kT}\right\},$$

i.e.,

$$\boxed{Z_i(T) = \frac{\exp(-\hbar\omega_i/2kT)}{1-\exp(-\hbar\omega_i/kT)}\,.} \tag{10.3.5}$$

The mean number $\langle n_i(T)\rangle$ is also useful and is defined by

$$\langle n_i(T)\rangle = \langle \Gamma_i^+\Gamma_i\rangle = \sum_n n\exp\left[-\frac{(n+\frac{1}{2})\hbar\omega_i}{kT}\right]\Big/Z_i(T)\,,$$

i.e.,

$$\boxed{\langle n_i(T)\rangle = [\exp(\hbar\omega_i/kT) - 1]^{-1}\,.} \tag{10.3.6}$$

Further items of use are the bath correlation functions. To obtain these we define

$$\Gamma_i(t) = \exp\left(\frac{\mathrm{i}\,H_\mathrm{B}t}{\hbar}\right)\Gamma_i\exp\left(-\frac{\mathrm{i}\,H_\mathrm{B}t}{\hbar}\right) = \exp(-\,i\omega_i t)\,\Gamma_i \tag{10.3.7}$$

so that the bath correlations are

$$\boxed{\begin{aligned}\langle \Gamma_i^+(t)\Gamma_j\rangle &= \mathrm{e}^{\,\mathrm{i}\omega_i t}\,\langle \Gamma_i^+\Gamma_j\rangle = \mathrm{e}^{\,\mathrm{i}\omega_i t}\,\delta_{ij}\langle n_i(T)\rangle\\ \langle \Gamma_i(t)\Gamma_j\rangle &= \langle \Gamma_i^+(t)\Gamma_j^+\rangle = 0\\ \langle \Gamma_i(t)\Gamma_j^+\rangle &= \mathrm{e}^{-\mathrm{i}\omega_i t}\,\langle \Gamma_j^+\Gamma_i + \delta_{ij}\rangle = \mathrm{e}^{-\mathrm{i}\omega_i t}\,\delta_{ij}[\langle n_i(T)\rangle + 1]\,.\end{aligned}} \tag{10.3.8}$$

10.3.2 Correlations of Smooth Functions of Bath Operators

Consider a variable

$$y(t) = \sum s_i\Gamma_i(t)\,\mathrm{e}^{\mathrm{i}\omega_0 t}\,. \tag{10.3.9}$$

(The factor exp $(\mathrm{i}\omega_0 t)$ is inserted with a view to the application in Sect. 10.4.2).

$$\begin{aligned}&\langle y(t)\rangle = 0\\ &\langle y(t)y(0)\rangle = \langle y^+(t)y^+(0)\rangle = 0\\ &\langle y^+(t)y(0)\rangle = \sum_i |s_i|^2\langle n_i(T)\rangle\exp[i(\omega_i-\omega_0)t]\\ &\langle y(t)y^+(0)\rangle = \sum_i |s_i|^2\langle n_i(T)+1\rangle\exp[-\,i(\omega_i-\omega_0)t].\end{aligned} \tag{10.3.10}$$

Suppose now that the modes i are spaced closely together so that ω_i is a smooth function of i.

Then we relate the variables thus.

$$s_i \to s(\omega_i) \to s(\omega) \tag{10.3.11}$$

$$\langle n_i(T)\rangle \to \langle n(\omega_i, T)\rangle \to \langle n(\omega, T)\rangle \tag{10.3.12}$$

and

$$\sum_i |s_i|^2 (\,.\,.\,) \to \int d\omega\, S(\omega)\,(\ldots)\,. \tag{10.3.13}$$

For sufficiently smooth functions $\langle n(\omega, T)\rangle$, $S(\omega)$, the correlation functions are rapidly exponentially decaying functions of t. For example, consider

$$\langle y^+(t)y(0)\rangle = \int_0^\infty d\omega\, S(\omega)\langle n(\omega, T)\rangle \exp[\mathrm{i}(\omega - \omega_0)t] \tag{10.3.14}$$

If, as is usual, $S(\omega)\langle n(\omega, T)\rangle$ is a smooth function of ω, then its fourier transform is a correspondingly rapidly decaying function of t. Thus, the correlation function (10.3.14) will be of the form of a rapidly decaying function of t, multiplied by $\exp(-\mathrm{i}\omega_0 t)$.

This is reminiscent of the relationship between correlation function and spectrum presented in Sect. 1.4.2, but here we have $\exp[\mathrm{i}(\omega - \omega_0)t]$ rather than $\cos \omega t$, and the correlation function is complex.

10.3.3 Quantum Master Equation for a System Interacting with a Heat Bath

We shall now show that when a quantum system interacts with a heat bath, we can apply the adiabatic elimination methods of Sect. 6.6 to develop a quantum master equation. The Laplace transform method used here is not the most usual, but it gives the answer very quickly and quite precisely.

We suppose the system is described by operators A, A^+, which can obey any commutation relations, and the bath by harmonic oscillator operators Γ_i, Γ_i^+.

The Hamiltonian is considered to be able to be written

$$H = \gamma^2 H_1 + \gamma H_2 + H_3\,. \tag{10.3.15}$$

H_3 is a function of A, A^+ only. H_2 can be written

$$H_2 = \sum_i (C g_i \Gamma_i^+ + C^+ g_i^* \Gamma_i) \tag{10.3.16}$$

where C is a function of A and A^+. Here g_i may be a function of time. H_1 is the bath Hamiltonian and has the form (10.3.1), though even this is not absolutely essential. The system and bath operators commute, and each acts in its own space. Thus, if H_2 is set equal to zero,

$$\rho = \rho_B \times \rho_s \tag{10.3.17}$$

and

$$\begin{aligned} i\hbar \frac{\partial \rho_B}{\partial t} &= \gamma^2 [H_1, \rho_B] \\ i\hbar \frac{\partial \rho_s}{\partial t} &= [H_3, \rho_s] \end{aligned} \tag{10.3.18}$$

and strictly speaking, we should write

$$\begin{aligned} A &= 1 \times \hat{A} \\ \Gamma_i &= \hat{\Gamma}_i \times 1 \end{aligned} \tag{10.3.19}$$

to indicate that each operates in its own space. The parameter γ is introduced to formalise the fact that the procedure is valid when the bath variables have a much faster time scale than the system variables, and that the g_i, to give a finite limit, must be considered to become large also. The exact implementation of this limit in a practical case depends on the knowledge of suitable variables which become large.

Then the equation of motion for the density matrix can be written

$$\frac{\partial \rho}{\partial t} = (\gamma^2 L_1 + \gamma L_2 + L_3)\rho \tag{10.3.20}$$

where

$$L_i \rho = -\frac{i}{\hbar}[H_i, \rho]\,, \tag{10.3.21}$$

The operators L_i are known as *Liouville operators,* and are linear operators. Equation (10.3.20) is now in exactly the right form for us to apply adiabatic elimination techniques, provided we can define a suitable projector. We choose a projector of the following kind:

$$\boxed{P\rho = \rho(T) \times \mathrm{Tr}_B\{\rho\} \equiv \rho(T) \times \hat{\rho}\,,} \tag{10.3.22}$$

where $\rho(T)$ is defined by (10.3.2), and by Tr_B we mean the trace only over the bath states. That is, a complete set of states for the systems and bath can be written

$|n_B, n_s\rangle$ and

$$\mathrm{Tr}_{\mathrm{B}}\{A\} = \sum_{\mathrm{B}} \langle n_{\mathrm{B}}, n_{\mathrm{s}} | A | n_{\mathrm{B}}, n_{\mathrm{s}} \rangle\,. \tag{10.3.23}$$

P is obviously a projector and

$$L_1 P\rho = -\frac{\mathrm{i}}{h}[H_1, \rho(T) \times \mathrm{Tr}_{\mathrm{B}}\rho] = 0 \tag{10.3.24}$$

since H_1 operates only on bath states. Also

$$PL_1\rho = -\frac{\mathrm{i}}{\hbar}\rho(T) \times \mathrm{Tr}_{\mathrm{B}}\{[H_1, \rho]\} = 0 \tag{10.3.25}$$

so the requirement

$$PL_1 = L_1 P = 0 \tag{10.3.26}$$

is satisfied.

We now check whether PL_2P vanishes, i.e., what is

$$-\frac{\mathrm{i}}{\hbar}\rho(T) \times \mathrm{Tr}_{\mathrm{B}}\{[H_2, \rho(T) \times \hat{\rho}]\}\,. \tag{10.3.27}$$

If we substitute a typical term from H_2, namely, $C\Gamma_i^{\dagger}$ into this we get

$$-\frac{\mathrm{i}}{\hbar}\rho(T) \times \mathrm{Tr}_{\mathrm{B}}\{\Gamma_i^{\dagger}\rho(T) \times C\hat{\rho} - \rho(T)\Gamma_i^{\dagger} \times \hat{\rho}C\} \tag{10.3.28}$$

and obviously

$$\mathrm{Tr}_{\mathrm{B}}\{\Gamma_i^{\dagger}\rho(T)\} = \mathrm{Tr}_{\mathrm{B}}\{\rho(T)\Gamma_i^{\dagger}\} = 0\,. \tag{10.3.29}$$

Similarly, all other terms vanish.

We can therefore carry out the same adiabatic elimination procedure as in Sect. 6.6 to obtain

$$\frac{\partial}{\partial t}[\rho(T) \times \hat{\rho}] = L_3\rho(T) \times \hat{\rho} - PL_2L_1^{-1}L_2\rho(T) \times \hat{\rho} \tag{10.3.30}$$

which reduces to

$$\boxed{\frac{\partial\hat{\rho}}{\partial t} = L_3\hat{\rho} - \mathrm{Tr}_{\mathrm{B}}\{L_2L_1^{-1}L_2\rho(T) \times \hat{\rho}\}\,.} \tag{10.3.31}$$

This can now be reduced explicitly to a simpler form. However, we must first deal with L_1^{-1}. In Sect. 6.6 we used the relation

$$L_1^{-1}(1 - P) = \int_0^\infty e^{L_1 t} dt (1 - P) \tag{10.3.32}$$

whose truth depended on the truth of

$$\lim_{t\to\infty} e^{L_1 t}(1 - P) = 0$$

which arose from the assumption that L_1 had negative eigenvalues.

Such a strong result is not in fact needed here — all that is needed is that the particular traces of Γ_i, Γ_i^+ over the bath in the stationary density operator $\rho(T)$ vanish at large t and this will occur if the particular linear combinations which turn up, namely, $\sum g_i \Gamma_i^+$, etc, have coefficients g_i which satisfy the smoothness conditions for $S(\omega)$ discussed in Sect. 10.3.2. However, we leave the detailed checking of this to the reader and adopt (10.3.32).

We note that for any operator A, we can define

$$A(t) \equiv e^{-L_1 t} A = \exp(\mathrm{i} H_1 t/\hbar) A \exp(-\mathrm{i} H_1 t/\hbar) \tag{10.3.33}$$

which is a notation in agreement with (10.3.7). A proof is obtained by explicit differentiation and definition of L_1:

$$\frac{dA(t)}{dt} = \frac{\mathrm{i}}{\hbar}[H_1, A(t)] = -L_1 A(t)\,. \tag{10.3.34}$$

We can now proceed:

$$\begin{aligned}
&-\mathrm{Tr}_B\{L_2 L_1^{-1} L_2 \rho(T) \times \hat{\rho}\} \\
&\quad = \frac{1}{\hbar^2}\int_0^\infty \mathrm{Tr}_B\{[H_2, e^{-\mathrm{i}H_1 t/\hbar}[H_2, \rho(T) \times \hat{\rho}]e^{\mathrm{i}H_1 t/\hbar}]\}\, dt \\
&\quad = \frac{1}{\hbar^2}\int_0^\infty \mathrm{Tr}_B\{[e^{H_1 t/\hbar} H_2 e^{-\mathrm{i}H_1 t/\hbar}, [H_2, \rho(T) \times \hat{\rho}]]\}\, dt \qquad (10.3.35)\\
&\quad = \frac{1}{\hbar^2}\int_0^\infty \mathrm{Tr}_B\{\sum_{i,j}[C g_i \Gamma_i^+(t) + C^+ g_i^* \Gamma_i(t), [C g_j \Gamma_j^+ + C^+ g_j^* \Gamma_j, \rho(T) \times \hat{\rho}]]\}\, dt.
\end{aligned}$$

There are 16 terms in this expression. However, only those involving a Γ and a Γ^+ will be nonzero. Consider the particular term

$$\frac{1}{\hbar^2}\int_0^\infty \mathrm{Tr}_B\{\sum_{i,j} C g_i \Gamma_i^+(t) C^+ g_j^* \Gamma_j \rho(T) \times \hat{\rho}\}\, dt$$

which, with the identification $g_i^* \to S_i$, $\omega_0 \to 0$ from (10.3.9), we can write in terms of

$$y(t) = \sum g_i^* \Gamma_j(t)\,. \tag{10.3.36}$$

We introduce a notation similar to that normally used in quantum optics, namely,

$$\begin{aligned}
&\frac{1}{\hbar^2}\int_0^\infty dt\,\langle y^+(t)y(0)\rangle = \tfrac{1}{2}\bar{N}K + \mathrm{i}\delta \\
&\frac{1}{\hbar^2}\int_0^\infty dt\,\langle y^+(0)y(t)\rangle = \tfrac{1}{2}K\bar{N} - \mathrm{i}\delta \\
&\frac{1}{\hbar^2}\int_0^\infty dt\,\langle y(t)y^+(0)\rangle = \tfrac{1}{2}K(\bar{N}+1) - \mathrm{i}\delta' \\
&\frac{1}{\hbar^2}\int_0^\infty dt\,\langle y(0)y^+(t)\rangle = \tfrac{1}{2}K(\bar{N}+1) + \mathrm{i}\delta' \,.
\end{aligned} \tag{10.3.37}$$

The double commutators can be evaluated, and we find that the Master equation reduces to

$$\boxed{\begin{aligned}
\frac{\partial\hat{\rho}}{\partial t} = {}& -\frac{\mathrm{i}}{h}[H_3, \hat{\rho}] - \mathrm{i}\delta[C^+C, \hat{\rho}] + \mathrm{i}\delta'[CC^+, \hat{\rho}] \\
& \tfrac{1}{2}K(1+\bar{N})\,(2C\hat{\rho}C^+ - C^+C\hat{\rho} - \hat{\rho}C^+C) \\
& + \tfrac{1}{2}K\bar{N}(2C^+\hat{\rho}C - CC^+\hat{\rho} - \hat{\rho}CC^+) \equiv \hat{L}\hat{\rho}.
\end{aligned}} \tag{10.3.38}$$

Comments

i) The Master equation in the above form is conventionally derived by methods, not using the Laplace transform, but using essentially only time domain equations. See, for example, the work by *Louisell* [10.5]. Our derivation relies on L_3 being very much less than $\gamma^2 L_1$ and, in practice, in optical situations this is not so, since H_3 represents a rapid oscillation which decays quite slowly as a result of the interaction with the heat bath. This can usually be taken account of by introducing the *interaction picture* and the result is a Master equation of the form (10.3.38) in which explicit time dependence of g_i occurs and ω_0 in (10.3.9) is the natural frequency of the free motion. This procedure is demonstrated in our treatment of the two level atom in Sect. 10.4.2. It is not our aim to give a full and detailed account here, and the reader is referred to *Louisell*'s work for a full explanation.

ii) The Master equation (10.3.38) can be regarded as the *definition* of a quantum Markov process, which is simpler from a mathematical point of view. The quantities H_3 and C specifying the free motion and the interaction with the heat bath are quite arbitrary, as is their relationship to each other. Notice, however, that the definition is incomplete without a specification of how to define multitime averages; in other words, to define an analogue of the multitime joint probabilities which are basic to ordinary stochastic processes.

iii) In a zero temperature bath, $\rho_B = |0\rangle\langle 0|$ and all averages of $\Gamma^+\Gamma$ vanish. This means that

$$T = 0 \Longrightarrow \bar{N} = 0 \tag{10.3.39}$$

and the third line does not contribute. This line, therefore, is related to thermal noise.

10.4 Examples and Applications of Quantum Markov Processes

10.4.1 Harmonic Oscillator

Here we take

$$\begin{aligned} &C \to a \\ &C^+ \to a^+ \\ &[a, a^+] = 1 \\ &H_3 = \hbar\omega(a^+a + \tfrac{1}{2}) \,. \end{aligned} \tag{10.4.1}$$

Using the commutation relations, we find that (10.3.38) reduces to (where we now write simply ρ instead of $\hat{\rho}$.)

$$\begin{aligned} \frac{\partial\rho}{\partial t} = &- \mathrm{i}\omega'[a^+a, \rho] \\ &+ \tfrac{1}{2}K(\bar{N} + 1)\,(2a\rho a^+ - a^+a\rho - \rho a^+a) \\ &+ \tfrac{1}{2}K\bar{N}(2a^+\rho a - aa^+\rho - \rho aa^+) \,, \end{aligned} \tag{10.4.2}$$

where

$$\omega' = \omega + \delta - \delta' \,.$$

a) Diagonal Matrix Elements

The diagonal matrix element

$$\langle n|\rho|n\rangle \equiv P(n) \tag{10.4.3}$$

represents the probability of there being n quanta in the system. We easily check that (using the properties of a^+ and a defined in Sect. 10.1)

$$\begin{aligned} \partial_t P(n) = {} & K(\bar{N} + 1)[(n + 1)P(n + 1) - nP(n)] \\ & + K\bar{N}[nP(n-1) - (n + 1)P(n)] \,. \end{aligned} \tag{10.4.4}$$

This is an ordinary birth-death Master equation. Notice that the transition probabilities have the form

$$\begin{aligned} t^+(n) &= K\bar{N}(n + 1) \\ t^-(n) &= K(\bar{N} + 1)n \end{aligned} \tag{10.4.5}$$

so that the probability of creating a quantum has a part proportional to $(n + 1)$. A chemical reaction of the form

$$\begin{aligned} &B \rightleftharpoons X \\ &A + X \to 2X \end{aligned} \tag{10.4.6}$$

would have a similar Master equation.

The solution in the stationary state is

$$P_s(n) = \left(\frac{\bar{N}}{1+\bar{N}}\right)^n \frac{1}{1+\bar{N}}. \tag{10.4.7}$$

This is the usual Boltzmann distribution, in which one can identify

$$\frac{\bar{N}}{1+\bar{N}} = \exp(-\hbar\omega/kT) \tag{10.4.8}$$

which means

$$\bar{N} = 1/[\exp(\hbar\omega/kT) - 1] \tag{10.4.9}$$

which determines $\bar{N}$ in terms of T, or conversely. We note that

$$\begin{aligned} \langle n \rangle_s &= \bar{N} \\ \operatorname{var}\{n\}_s &= \bar{N}^2 \,. \end{aligned} \tag{10.4.10}$$

b) Fokker-Planck Equation from P-Representation

We use the Glauber-Sudarshan P-representation from Sect. 10.2.2 and the operator correspondence from (10.2.26) of Sect. 10.2.3. Remembering that operator products written on the right of ρ are reversed in order when the correspondence is made, we derive the Fokker-Planck equation for the P function:

$$\boxed{\frac{\partial P}{\partial t} = \left[\tfrac{1}{2}K\left(\frac{\partial}{\partial \alpha}\alpha + \frac{\partial}{\partial \alpha^*}\alpha^*\right) - \mathrm{i}\omega'\left(\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^*\right) + K\bar{N}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P} \tag{10.4.11}$$

This is a form of complex Ornstein-Uhlenbeck process. For we can write

$$\alpha = x + \mathrm{i}y$$

and

$$\partial_\alpha = \tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y)$$

and get

$$\boxed{\frac{\partial P}{\partial t} = \tfrac{1}{2}K\left[\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right) + \omega'\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) + \frac{\bar{N}}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]P\,.} \tag{10.4.12}$$

This result is very simple compared to the Poisson representation Fokker-Planck equations which always involve nonconstant diffusion terms. The stochastic differential equations equivalent to (10.4.12) are

$$\boxed{\begin{aligned} dx &= -(\tfrac{1}{2}Kx + \omega y)dt + \sqrt{\tfrac{1}{2}\bar{N}}\, dW_1(t) \\ dy &= -(\tfrac{1}{2}Ky - \omega x)dt + \sqrt{\tfrac{1}{2}\bar{N}}\, dW_2(t)\,. \end{aligned}} \tag{10.4.13}$$

These represent the equations for a damped oscillator, very like those considered in Sect. 5.3.6d.

These are often written as one complex Langevin equation:

$$d\alpha = -\,(\tfrac{1}{2}K - \mathrm{i}\omega)\alpha\, dt + \sqrt{\bar{N}}\, d\eta(t)\,, \tag{10.4.14}$$

where $d\eta(t)$ is the increment of a complex Wiener process satisfying

$$\begin{aligned} &\langle d\eta(t)\rangle = \langle d\eta^*(t)\rangle = \langle d\eta(t)\; d\eta(t')\rangle = \langle d\eta^*(t)\; d\eta^*(t')\rangle = 0 \\ &\langle d\eta(t)\; d\eta^*(t)\rangle = dt \end{aligned} \tag{10.4.15}$$

and explicitly given by

$$d\eta(t) = [dW_1(t) + \mathrm{i}dW_2(t)]/\sqrt{2}\,. \tag{10.4.16}$$

The use of the complex $d\eta(t)$ is not very easily generalised and we will not use them any further. However, the complex variable FPE is useful, and will be maintained.

The solutions of the Ornstein-Uhlenbeck process are given in Sect. 4.4.6. We find

$$\begin{aligned} &\langle \alpha(t)\rangle = \alpha(0)\exp[-(K/2 + \mathrm{i}\omega)t] \\ &\langle \alpha^*(t)\alpha(t)\rangle = \langle \alpha^*(0)\alpha(0)\rangle\, \mathrm{e}^{-Kt} + \bar{N}(1 - \mathrm{e}^{-Kt}) \end{aligned} \tag{10.4.17}$$

$$\begin{aligned} &\langle \alpha^2\rangle_\mathrm{s} = \langle \alpha^{*2}\rangle_\mathrm{s} = 0 \\ &\langle \alpha\alpha^*\rangle_\mathrm{s} = \langle a^+a\rangle_\mathrm{s} = \bar{N}\,. \end{aligned} \tag{10.4.18}$$

Only when $\bar{N} = 0$ do we find that $\langle a^+a\rangle_\mathrm{s}$ vanishes, i.e., at zero temperature. The time correlation functions of this Ornstein-Uhlenbeck process require interpretation which will be done in Sect 10.5.

c) Inclusion of a Driving Field

Suppose we consider the driven damped harmonic oscillator obtained by choosing

$$H_3 = \hbar\omega(a^+a + \tfrac{1}{2}) + \hbar(\varepsilon a^+ + \varepsilon^* a) \tag{10.4.19}$$

which represents a quantised oscillator interacting with a nonquantised field (Sect. 10.1.1).

The FPE develops an extra term

$$\left(\mathrm{i}\varepsilon\,\frac{\partial}{\partial\alpha} - \mathrm{i}\varepsilon^*\,\frac{\partial}{\partial\alpha^*}\right) P \tag{10.4.20}$$

which allows the FPE to be written.

$$\frac{\partial P}{\partial t}=\Big[(\tfrac{1}{2}K-\mathrm{i}\omega')\frac{\partial}{\partial\alpha}\Big(\alpha+\frac{\mathrm{i}\varepsilon}{\frac{1}{2}K-\mathrm{i}\omega'}\Big)$$

$$+(\tfrac{1}{2}K+\mathrm{i}\omega')\frac{\partial}{\partial\alpha^*}\Big(\alpha^*-\frac{\mathrm{i}\varepsilon}{\frac{1}{2}K+\mathrm{i}\omega'}\Big)+K\bar{N}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\Big]P \tag{10.4.21}$$

which again represents an Ornstein-Uhlenbeck process, but with the origin shifted so that

$$\langle\alpha\rangle_s=-\mathrm{i}\varepsilon/(\tfrac{1}{2}K-\mathrm{i}\omega')=\langle a\rangle_s \tag{10.4.22}$$

$$\begin{aligned}&\langle\alpha\alpha^*\rangle-\langle\alpha\rangle\langle\alpha^*\rangle=\bar{N}\\&\langle\alpha^2\rangle-\langle\alpha\rangle^2=\langle\alpha^{*2}\rangle-\langle\alpha^*\rangle^2=0\end{aligned} \tag{10.4.23}$$

and the time-dependent solution for the mean is

$$\langle\alpha(t)\rangle=\langle\alpha(0)\rangle\exp\Big[-\Big(\frac{K}{2}-\mathrm{i}\omega'\Big)t\Big]$$

$$-\frac{\mathrm{i}\varepsilon}{\frac{1}{2}K+\mathrm{i}\omega'}\Big\{1-\exp\Big[-\Big(\frac{K}{2}-\mathrm{i}\omega'\Big)t\Big]\Big\}. \tag{10.4.24}$$

The stationary distribution in the P-representation is a Gaussian with variance $\bar{N}$ and mean given by (10.4.22). If $\bar{N}$ is small, this represents a density matrix ρ corresponding almost to that of the pure coherent state $|\alpha_s\rangle\langle\alpha_s|$. Thus, a good approximation to a coherent state is provided by a classical driving field, interacting with a harmonic oscillator which interacts with a low temperature bath.

d) Driving by a Fluctuating Field

Suppose in (10.4.19) ε is a function of time $\varepsilon(t)$, possibly stochastic. The Langevin equation for α, α^* is

$$d\alpha(t)=[-(\tfrac{1}{2}K-\mathrm{i}\omega')\alpha(t)-\mathrm{i}\varepsilon(t)]dt+\sqrt{\frac{K\bar{N}}{2}}[dW_1(t)+\mathrm{i}dW_2(t)]\,. \tag{10.4.25}$$

Since $\varepsilon(t)$ occurs linearly and multiplied by a constant, we can make some simple stochastic assumptions.

The simplest is to take the short correlation time limit and replace

$$\varepsilon(t)dt=\varepsilon_0\,dt+\sqrt{\frac{f}{2}}[dW_3(t)+\mathrm{i}dW_4(t)]$$

which gives independent fluctuations in each component [since the coefficient of $\varepsilon(t)$ is constant, there is no Ito-Stratonovich ambiguity]. The net effect is to modify (10.4.21) by the replacement

$$K\bar{N}\to K\bar{N}+f\,, \tag{10.4.26}$$

that is, the extra noise simply increases the thermal noise already present.

Alternatively, equations of motion for $\varepsilon(t)$ may be assumed and solved in conjunction with the stochastic differential equation (10.4.25),

10.4.2. The Driven Two-Level Atom

An idealisation of atomic systems is the *two-level atom* which may exist in either of two states. We set up a matrix formalism with the matrices

$$\begin{aligned} |-\rangle &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{ground state} \\ |+\rangle &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{excited state.} \end{aligned} \tag{10.4.27}$$

The system can make transitions which are described by the Pauli matrices

$$S^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad S_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{10.4.28}$$

Thus, S^+ lifts the atom to the excited state, S^- drops it to the ground state. The Hamiltonian for the system in interaction with a radiation field and a driving field $E(t)$ can be written as

$$\begin{aligned} H_3 &= \tfrac{1}{2}\hbar\omega_0 S_z - \tfrac{1}{2}\hbar d[S^+E(t) + S^-E^*(t)] \\ H_2 &= \sum_i [g_i S^+ a_i + g_i{}^* S^- a_i^+] \\ H_1 &= \sum_i \hbar\omega_i(a_i^+ a_i + \tfrac{1}{2}) . \end{aligned} \tag{10.4.29}$$

Physically, the term $\frac{1}{2}\hbar\omega_0 S_z$ assigns energy $\frac{1}{2}\hbar\omega_0$ to the excited state, $-\frac{1}{2}\hbar\omega_0$ to the lower state, the energy of excitation thus being $\hbar\omega_0$. The second term in H_2 represents the coupling through a dipole moment d of the driving field to the atom and is of essentially the same form as H_3, in which, however, all the other modes i are quantised. The modes i are to be thought of as representing photons of energy $\hbar\omega_0$ travelling in various different directions labelled by i, whereas the driving field, which is a strong classical field, is in a single particular direction.

The quanta of the radiation field provide the heat bath. Thus we use a_i for Γ_i. The heat bath can, in principle, be at any temperature, but is normally at essentially zero temperature in experimental situations. We shall preserve a nonzero temperature T.

The driving field $E(t)$ is assumed to have the same frequency ω_0 as that required to excite the atom: thus, we write

$$E(t) = \varepsilon\, e^{-i\omega_0 t} . \tag{10.4.30}$$

It is best here to work in the *interaction picture* which removes the rapid oscillations. Define

$$\tilde{\rho}(t) = \exp\left(\frac{i\omega_0}{2} S_z t\right) \rho \exp\left(-\frac{i\omega_0}{2} S_z t\right). \tag{10.4.31}$$

Then we can describe the motion of $\tilde{\rho}(t)$ by the Hamiltonians

$$\begin{aligned}
\tilde{H}_3 &= -\tfrac{1}{2}\hbar d(S^+\varepsilon + S^-\varepsilon^*) \\
\tilde{H}_2 &= \sum_l (g_l e^{i\omega_0 t} S^+ a_l + g_l^* e^{-i\omega_0 t} S^- a_l^+) \\
\tilde{H}_1 &= \sum_l \hbar\omega_l (a_l^+ a_l + \tfrac{1}{2}) .
\end{aligned} \tag{10.4.32}$$

The reader may check that all derivations go through in exactly the same way as in Sect. 10.3.3, with the difference that we make the the replacement

$$g_l \to g_l e^{i\omega_0 t} \tag{10.4.33}$$

which means that the $y(t)$ are now defined *exactly* as in (10.3.9). Thus, the explicit exponential time dependence does arise here as a result of the free motion of the atom.

The master equation becomes, for $\hat{\rho}$, the reduced density matrix (in the interaction picture), after using the algebra corresponding to the explicit forms of the matrices (10.4.28),

$$\begin{aligned}
\frac{\partial \hat{\rho}}{\partial t} &= \tfrac{1}{2} i d[S^+\varepsilon^* + S^-\varepsilon, \hat{\rho}] \\
&\quad + \tfrac{1}{2}K(1 + \bar{N})[2S^-\hat{\rho}S^+ - \tfrac{1}{2}(1 + S_z)\hat{\rho} - \tfrac{1}{2}\hat{\rho}(1 + S_z)] \\
&\quad + \tfrac{1}{2}K\bar{N}[2S^+\hat{\rho}S^- - \tfrac{1}{2}(1 - S_z)\hat{\rho} - \tfrac{1}{2}\hat{\rho}(1 - S_z)] .
\end{aligned} \tag{10.4.34}$$

We cannot use coherent states here, since the S^+, S_z operate in a two-dimensional space and are not harmonic oscillator operators. The density matrix is a 2 × 2 matrix with unit trace and is completely specified by the quantities $\mathrm{Tr}\{\rho S^\pm\}$, $\mathrm{Tr}\{\rho S_z\}$, which are the expectations of the operators. Hence, we derive as a complete specification of $\rho(t)$,

$$\begin{aligned}
\frac{d}{dt}\langle S^+\rangle &= -\tfrac{1}{2}K(2\bar{N} + 1)\langle S^+\rangle + i d\varepsilon\langle S_z\rangle \\
\frac{d}{dt}\langle S^-\rangle &= -\tfrac{1}{2}K(2\bar{N} + 1)\langle S^-\rangle - i d\varepsilon^*\langle S_z\rangle \\
\frac{d}{dt}\langle S_z\rangle &= -K(2\bar{N} + 1)\langle S_z\rangle - K + \tfrac{1}{2} i d(\varepsilon^*\langle S^+\rangle - \varepsilon\langle S^-\rangle) .
\end{aligned} \tag{10.4.35}$$

These equations have been investigated in detail by *Carmichael* and *Walls* [10.6]. We note a few points.

a) Stationary Solution with $\varepsilon = 0$

Here,

$$\begin{aligned}\langle S^+\rangle &= \langle S^-\rangle = 0\\ \langle S_z\rangle &= -\frac{1}{2\bar{N}+1}\,.\end{aligned} \qquad (10.4.36)$$

Setting $\bar{N} = (e^{\hbar\omega/kT} - 1)^{-1}$ gives the correct Boltzmann distribution.

b) Temperature Dependent Relaxation

In contrast to the harmonic oscillator, the relaxation rate is proportional to $2\bar{N} + 1$ which depends on temperature. The difference in operator algebra leads to this.

c) Comparison with Bath Temperature

If we explicitly evaluate the quantities $K, \bar{N}, \delta, \delta'$, using (10.3.12) and related forms, we find

$$\tfrac{1}{2}K\bar{N} + i\delta = \hbar^2 \int_0^\infty dt\, e^{-i\omega_0 t} \int_0^\infty d\omega\, S(\omega)\, \langle n(\omega, T)\rangle\, e^{i\omega t}\,. \qquad (10.4.37)$$

We note the identity

$$\int d\Omega \int_0^\infty f(\Omega) e^{\pm i\Omega t} = \pi f(0) \pm i \fint d\Omega\, f(\Omega)/\Omega\,, \qquad (10.4.38)$$

where $\fint$ represents the Cauchy principal value of an integral containing $1/\Omega$. Hence,

$$\tfrac{1}{2}K\bar{N} + i\delta = \pi h^2 S(\omega_0)\, \langle n(\omega_0, T)\rangle + i\hbar^2 \fint_0^\infty d\omega\, S(\omega)\langle n(\omega, T)\rangle(\omega-\omega_0)^{-1}. \qquad (10.4.39)$$

Similarly,

$$\begin{aligned}\tfrac{1}{2}K(\bar{N} + 1) + i\delta' &= \pi\hbar^2 S(\omega_0)\langle n(\omega_0, T) + 1\rangle\\ &\quad + i\hbar^2 \fint_0^0 d\omega\, S(\omega)\langle n(\omega, T) + 1\rangle(\omega - \omega_0)^{-1}\end{aligned} \qquad (10.4.40)$$

which is the more usual form for the damping constants.

From these equations, one notes

$$\boxed{\begin{aligned}K &= 2\pi\hbar^2 S(\omega_0)\\ \bar{N} &= \langle n(\omega_0, T)\rangle = [\exp(\hbar\omega_0/kT) - 1]^{-1}\\ -\delta + \delta' &= \hbar^2 \fint_0^\infty d\omega\, S(\omega)\, (\omega - \omega_0)^{-1}\,.\end{aligned}} \qquad \begin{aligned}&(10.4.41)\\ &(10.4.42)\\ &(10.4.43)\end{aligned}$$

Notice that (10.4.42) is consistent with (10.4.36), i.e., the atom comes to the same temperature as the bath.

d) Stationary Mean in Nonzero Field

We can solve (10.4.35) to obtain

$$\langle S_z \rangle = \frac{-K^2(2\bar{N}+1)}{K^2(2\bar{N}+1)^2 + 2d^2|\varepsilon|^2} \tag{10.4.44}$$

$$\langle S^+ \rangle = \langle S^- \rangle^* = \frac{-2\mathrm{i}dK\varepsilon}{K^2(2\bar{N}+1)^2 + 2d^2|\varepsilon|^2}\,. \tag{10.4.45}$$

e) Connection with the Random Telegraph Process: the Einstein Equations

When $E(t) = 0$, we may define

$$\begin{aligned} P(+) &= \langle + | \rho | + \rangle \\ P(-) &= \langle - | \rho | - \rangle \end{aligned} \tag{10.4.46}$$

and derive the Master equation for $P(+)$ and $P(-)$.

$$\boxed{\begin{aligned} \partial_t P(+) &= -K(1+\bar{N})P(+) + K\bar{N}P(-) \\ \partial_t P(-) &= K(1+\bar{N})P(+) - KNP(-)\,. \end{aligned}} \tag{10.4.47}$$

These are identical to the equations for the random telegraph process of Sect. 3.8.5 Thus, the two-level atom can be regarded as a quantum random telegraph process. However, if $E(t) \neq 0$, the off-diagonal matrix elements are also involved, giving a truly quantum process.

Equations (10.4.47) were first introduced by *Einstein* [10.7] and are thus nowadays known as the *Einstein equations*. They demonstrate two effects, namely,

i) *stimulated emission and absorption*: terms proportional to $\bar{N}$ depend on the number of photons in the radiation field and are called stimulated terms. Thus, the processes of excitation ($- \rightarrow +$) and de-excitation ($+ \rightarrow -$), with a corresponding absorption or emission of a photon, can occur as processes stimulated by the number of photons already in the field.

ii) *Spontaneous emission*: i.e., de-excitation ($+ \rightarrow -$) occurs by the term $KP(+)$ occurring in both equations. De-excitation can occur even though no photons are present. This is not surprising from a modern point of view, but was an important innovation when first introduced.

10.5 Time Correlation Functions in Quantum Markov Processes

For a quantum system the two-time correlation function $\langle A(t')B(t) \rangle$ can be easily defined exactly: we simply state the result here. If $\rho(t)$ is the density matrix at time t (in the Schrödinger picture), H is the Hamiltonian and A and B are the operators for the variables to be measured, then

$$\langle A(t)\rangle = \mathrm{Tr}\{A\rho(t)\} \tag{10.5.1}$$

and

$$\langle A(t+\tau)B(t)\rangle = \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}H\tau/\hbar}A\,\mathrm{e}^{-\mathrm{i}H\tau/\hbar}B\rho(t)\}\ . \tag{10.5.2}$$

[The truth of (10.5.2) arises from the Heisenberg picture form, where it becomes

$$\langle A(t+\tau)B(t)\rangle = \mathrm{Tr}\{A_{\mathrm{H}}(t+\tau)B_{H}(t)\rho_{\mathrm{H}}\} \tag{10.5.3}$$

where the Heisenberg density matrix ρ_{H} is time independent.]

Equation (10.5.2) is exact, but not useful. In a quantum Markov system, we want to express everything in terms of the Liouvillian for the reduced system, in which heat bath variables have been traced out. When this has been done, we have effectively defined multitime joint probabilities which enable us to specify the quantum Markov process completely.

This can be achieved relatively simply. Supposing A and B are operators only in the system space and not in the heat bath space. Then we can rewrite (10.5.3) as

$$\langle A(t+\tau)B(t)\rangle = \mathrm{Tr}_{\mathrm{S}}\{A\,\mathrm{Tr}_{\mathrm{B}}\{\mathrm{e}^{-\mathrm{i}H\tau/\hbar}B\rho(t)\mathrm{e}^{\mathrm{i}H\tau/\hbar}\}\} \tag{10.5.4}$$

(since A is proportional to the identity in the bath space).

The equation of motion for the term

$$X(\tau, t) = \mathrm{e}^{-\mathrm{i}H\tau/\hbar}\,B\rho(t)\mathrm{e}^{\mathrm{i}H\tau/\hbar} \tag{10.5.5}$$

in terms of τ is

$$ih\partial_\tau X(\tau, t) = [H, X(\tau, t)] \tag{10.5.6}$$

and in exactly the same way as we derived (10.3.38), which can be written abstractly as

$$\partial_t\hat{\rho}(t) = \hat{L}\rho(t), \tag{10.5.7}$$

for

$$\hat{\rho} = \mathrm{Tr}_{\mathrm{B}}\{\rho\}, \tag{10.5.8}$$

we can derive

$$\partial_\tau[\mathrm{Tr}_{\mathrm{B}}\{X(\tau, t)\}] = \hat{L}[\mathrm{Tr}_{\mathrm{B}}\{X(\tau, t)\}] \tag{10.5.9}$$

so that (10.5.4) is equivalent, in this adiabatic limit, to

$$\boxed{\langle A(t+\tau)B(t)\rangle = \mathrm{Tr}_{\mathrm{s}}\{A\,\mathrm{e}^{\hat{L}\tau}B\hat{\rho}(t)\}\ .} \tag{10.5.10}$$

By using a similar procedure, general multitime correlations can be worked out, e.g.,

$$\boxed{\langle A(t_3)B(t_2)C(t_1)\rangle = \mathrm{Tr}_s\{A\, \mathrm{e}^{\mathcal{L}(t_3-t_2)}\, B\, \mathrm{e}^{\mathcal{L}(t_2-t_1)}\, C\hat{\rho}(t_1)\}} \tag{10.5.11}$$

(in both of these, $\hat{L}$ operates on everything to the right of it) and so on.

Since the operators do not all commute, the time ordering need not be the same as the operator ordering. This allows a wide range of possbilities, not all of which give simple formulae. A particular case that does turn up is the form

$$\begin{aligned}\langle A(t)B(t+\tau)C(t+\tau)D(t)\rangle &= \mathrm{Tr}\{A\, \mathrm{e}^{\mathrm{i}H\tau/\hbar}\, BC\, \mathrm{e}^{-\mathrm{i}H\tau/\hbar}\, D\rho(t)\} \\ &= \mathrm{Tr}_s\{BC\, \mathrm{Tr}_B\{\mathrm{e}^{-\mathrm{i}H\tau/\hbar}\, D\rho(t)A\, \mathrm{e}^{\mathrm{i}H\tau/\hbar}\}\}\end{aligned}$$

so that

$$\boxed{\langle A(t)B(t+\tau)C(t+\tau)D(t)\rangle = \mathrm{Tr}_s\{BC\, \mathrm{e}^{\mathcal{L}\tau}\, D\hat{\rho}(t)A\}\,.} \tag{10.5.12}$$

Note that (10.5.10) is a special case of this, as by setting $C(t) = D(t) = 1$, we find

$$\boxed{\langle A(t)B(t+\tau)\rangle = \mathrm{Tr}_s\{B\, \mathrm{e}^{\mathcal{L}\tau}\, \hat{\rho}(t)A\}} \tag{10.5.13}$$

The complete set of correlation functions for all possible operators defines all possible joint averages and may thus be taken as a definition of the Markov property in a quantum system in which $\hat{L}$ can be any operator which preserves $\mathrm{Tr}_s\{\hat{\rho}\}$ and the positivity of $\hat{\rho}$. In particular, $\hat{L}$ as defined in (10.3.8) can be used and is, in practice, the only kind of operator that does occur.

10.5.1 Quantum Regression Theorem

In the case where linear equations turn up for the means, we can develop a quantum regression theorem, similar to that for an ordinary Markov process (Sect. 3.7.4) This result was first derived by *Lax* [10.8].

Suppose for a certain set of operators Y_i, the Master equation can be shown to yield for any initial $\hat{\rho}$,

$$\partial_t\langle Y_i(t)\rangle = \sum G_{ij}(t)\langle Y_j(t)\rangle\,. \tag{10.5.14}$$

Then we assert that

$$\partial_\tau\langle Y_i(t+\tau)Y_l(t)\rangle = \sum_{i,j} G_{ij}(\tau)\langle Y_j(t+\tau)Y_l(t)\rangle\,. \tag{10.5.15}$$

For

$$\langle Y_i(t+\tau)Y_l(t)\rangle = \mathrm{Tr}_s\{Y_i \mathrm{e}^{\mathcal{L}\tau} Y_l\hat{\rho}(t)\} \tag{10.5.16}$$

and the right-hand side is an average of Y_t at time $t + \tau$ with the choice of initial density matrix

$$\rho_{\mathrm{init}} = Y_t \rho(t) . \tag{10.5.17}$$

Since by hypothesis any initial $\hat{\rho}$ is permitted and the hypothesis is linear, we may in fact, generate, any initial condition whatsoever. (It might be thought that the requirement that $\hat{\rho}$ be hermitian and positive semidefinite could limit the available initial conditions, but this is not so. For example, in two dimensions there are four linearly-independent, positive semidefinite matrices with unit trace, namely,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\mathrm{i} \\ -\frac{1}{2}\mathrm{i} & \frac{1}{2} \end{bmatrix}, \tag{10.5.18}$$

which provide a basis for the space of all 2 × 2 matrices. This property generalises straightforwardly to arbitrary dimensions). Hence, choosing ρ_{init} as defined in (10.5.17), the hypothesis (10.5.14) yields the result (10.5.15) which is the *quantum regression theorem.*

10.5.2 Application to Harmonic Oscillator in the P-Representation

We consider firstly the normally ordered time correlation function

$$\begin{aligned} \langle a^+(t + \tau)a(t)\rangle &= \mathrm{Tr}\{a^+ e^{\mathcal{L}\tau} a \int d^2\alpha\, P(\alpha, t) |\alpha\rangle \langle\alpha|\} \\ &= \mathrm{Tr}\{a^+ e^{\mathcal{L}\tau} \int d^2\alpha\, \alpha P(\alpha, t) |\alpha\rangle\langle\alpha|\} . \end{aligned} \tag{10.5.19}$$

The term

$$e^{\mathcal{L}\tau} \int d^2\alpha\, \alpha P(\alpha, t) |\alpha\rangle\langle\alpha|$$

is the solution of a generalised Master equation, for which the initial condition is the operator whose P-representation density is $\alpha P(\alpha, t)$. If a FPE corresponding to $\hat{L}$ exists [e.g., (10.4.11) (10.4.21)] this quantity is the density matrix with P function given by

$$\int d^2\alpha' \alpha' P(\alpha, t + \tau | \alpha', t) P(\alpha', t) \tag{10.5.20}$$

so that, using the cyclic property of the trace and $\langle\alpha| a^+ = \alpha^* \langle\alpha|$,

$$\begin{aligned} \langle a^+(t + \tau)a(t)\rangle &= \mathrm{Tr}\{a^+ \int d^2\alpha d^2\alpha' \alpha' P(\alpha, t + \tau | \alpha', t) P(\alpha', t) |\alpha\rangle\langle\alpha|\} \\ &= \int d^2\alpha d^2\alpha' \alpha^* \alpha' P(\alpha, t + \tau ; \alpha' t) , \end{aligned}$$

this means that

$$\boxed{\langle a^+(t + \tau)a(t)\rangle = \langle \alpha^*(t + \tau)\alpha(t)\rangle} \tag{10.5.21}$$

where $\alpha(t)$ is the P-representation stochastic variable. Similarly, for any function of a^+ and a,

$$\boxed{\langle F_1[a^+(t+\tau)]F_2[a(t)]\rangle = \langle F_1[\alpha^*(t+\tau)]F_2[\alpha(t)]\rangle\,.} \tag{10.5.22}$$

For non-normally ordered products, the answer is not so simple.

For example, consider

$$\langle a^+(t+\tau)a(t+\tau)a^+(t)a(t)\rangle = \mathrm{Tr}\{a^+a\, \mathrm{e}^{\mathcal{L}\tau}a^+a \int d^2\alpha P(\alpha,t)|\alpha\rangle\langle\alpha|\}\,; \tag{10.5.23}$$

using the operator identities of Sect. 10.2.3 (10.2.26)

$$= \mathrm{Tr}\left\{a^+a\, \mathrm{e}^{\mathcal{L}\tau} \int d^2\alpha\left[\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)\alpha P(\alpha,t)\right]|\alpha\rangle\langle\alpha|\right\} \tag{10.5.24}$$

and proceeding with the same argument as above,

$$= \mathrm{Tr}\left\{a^+a \int d^2\alpha \int d^2\alpha' P(\alpha,t+\tau|\alpha',t)\left[\left(\alpha'^* - \frac{\partial}{\partial\alpha'}\right)\alpha' P(\alpha',t)\right]|\alpha\rangle\langle\alpha|\right\} \tag{10.5.25}$$

and finally, using the same operator identities,

$$= \int d^2\alpha d^2\alpha'\left[\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)\alpha P(\alpha,t+\tau|\alpha',t)\right]\left[\left(\alpha'^* - \frac{\partial}{\partial\alpha'}\right)\alpha' P(\alpha',t)\right]. \tag{10.5.26}$$

To simplify, we may drop the $\partial/\partial\alpha$ since it integrates to zero, and integrate the $\partial/\partial\alpha'$ by parts, so that

$$(10.5.26) = \int d^2\alpha d^2\alpha'(\alpha^*\alpha)(\alpha'^*\alpha')P(\alpha,t+\tau;\alpha',t) \tag{10.5.27}$$

$$+ \int d^2\alpha' P(\alpha',t)\alpha' \frac{\partial}{\partial\alpha'}\left[\int d^2\alpha(\alpha^*\alpha)P(\alpha,t+\tau|\alpha',t)\right] \tag{10.5.28}$$

which can be rewritten as

$$\boxed{\begin{aligned}\langle a^+(t+\tau)a(t+\tau)a^+(t)a(t)\rangle &= \langle|\alpha(t+\tau)|^2|\alpha(t)|^2\rangle \\ &\quad + \langle\alpha' \frac{\partial}{\partial\alpha'}\langle|\alpha(t+\tau)|^2|[\alpha',t]\rangle\rangle\end{aligned}} \tag{10.5.29}$$

which is in close analogy to the Poisson representation result (7.7.76). In fact, if the P function depends only on $|\alpha|^2$ and $|\alpha'|^2$, they are identical, with the change from $|\alpha|^2$ (P-representation) to α (Poisson representation).

A measurement of more interest is the correlation function

$$\langle a^+(t)a^+(t+\tau)a(t+\tau)a(t)\rangle \equiv G^2(\tau,t) \tag{10.5.30}$$

which has been shown by *Glauber* [10.1] to be what is measured when one correlates intensities of light received at different times by a detector.

We use (10.5.12) to write

$$G^2(\tau, t) = \mathrm{Tr}\{a^+a\, e^{\mathcal{L}\tau} a\rho a^+\} \tag{10.5.31}$$

which, by similar reasoning to the above, yields

$$\boxed{\begin{aligned} G^2(\tau, t) &= \int d^2\alpha d^2\alpha' |\alpha^2| |\alpha'^2| P(\alpha, t+\tau; \alpha', t) \\ &= \langle |\alpha(t+\tau)|^2 |\alpha(t)|^2 \rangle\,. \end{aligned}} \tag{10.5.32}$$

This particularly simple result means that the measured intensity correlation function $G^2(\tau, t)$ is the same thing as the corresponding correlation function for the P-representation variables, and that it is *not* the same thing as (10.5.29),

Example: Driven Damped Oscillator. from Sect. 10.4.1c we have the stochastic differential equation for $\alpha - \alpha_s$:

$$d(\alpha - \alpha_s) = -(\tfrac{1}{2}K - i\omega')(\alpha - \alpha_s)dt + \sqrt{K\bar{N}}\, d\eta(t)$$

(with $d\eta^2 = d\eta^{*2} = 0$, $d\eta d\eta^* = dt$).
Hence, define

$$\tilde{I} = |(\alpha - \alpha_s)^2|$$

so that, using Ito's formula

$$d\tilde{I} = -K\tilde{I}dt + \sqrt{K\bar{N}}\,[(\alpha - \alpha_s)^* d\eta + (\alpha - \alpha_s)d\eta^*] + K\bar{N}dt$$

and

$$\frac{d\langle\tilde{I}\rangle}{dt} = -K(\langle\tilde{I}\rangle - \bar{N})\,.$$

Thus, $\langle\tilde{I}\rangle - \bar{N}$ obeys a linear equation and the quantum regression theorem can be applied, i.e., in the stationary state

$$\langle\tilde{I}(t)I(0)\rangle_s - \bar{N}^2 = e^{-Kt}(\langle\tilde{I}^2\rangle_s - \bar{N}^2)\,.$$

We use the Gaussian nature of the stationary distribution in $\alpha(t), \alpha^*(t)$ to note that

$$\langle\tilde{I}^2\rangle_s = \langle(x^2+y^2)^2\rangle = \langle x^4 + 2x^2y^2 + y^4\rangle \qquad (\text{here } \alpha - \alpha_s = x + iy)$$

and x and y are independent in the stationary state [from (10.4.23)] so, using (2.8.6),

$$\langle\tilde{I}^2\rangle_s = (\langle\tilde{I}\rangle_s)^2 = 2\bar{N}^2\,,$$

i.e.,

$$\boxed{\langle \tilde{I}(t)\tilde{I}(0)\rangle_s \equiv G^2(t)_s = \bar{N}^2(1+e^{-Kt})\,.} \tag{10.5.33}$$

More simply,

$$\boxed{\langle \alpha^*(t)\alpha(0)\rangle_s \equiv G^1(t)_s = \bar{N}\exp[-(\tfrac{1}{2}K + i\omega)t].} \tag{10.5.34}$$

10.5.3 Time Correlations for Two-Level Atom

These provide perfect applications of the regression theorem and are quickly deduced from Sect. 10.4.2. For simplicity, we consider only the stationary correlation functions:

$$\underset{\sim}{G}(t) = \begin{bmatrix} \langle S^+(t)S^+(0)\rangle_s & \langle S^+(t)S^-(0)\rangle_s & \langle S^+(t)S_z(0)\rangle_s \\ \langle S^-(t)S^+(0)\rangle_s & \langle S^-(t)S^-(0)\rangle_s & \langle S^-(t)S_z(0)\rangle_s \\ \langle S_z(t)S^+(0)\rangle_s & \langle S_z(t)S^-(0)\rangle_s & \langle S_z(t)S_z(0)\rangle_s \end{bmatrix}, \tag{10.5.35}$$

then

$$\frac{d\underset{\sim}{G}(t)}{dt} = \begin{bmatrix} -\tfrac{1}{2}K(2\bar{N}+1) & 0 & id\varepsilon \\ 0 & -\tfrac{1}{2}K(2\bar{N}+1) & -id\varepsilon^* \\ \tfrac{1}{2}id\varepsilon^* & -\tfrac{1}{2}id\varepsilon & -K(2\bar{N}+1) \end{bmatrix} \underset{\sim}{G}(t)\,. \tag{10.5.36}$$

This matrix equation can be integrated and the complete solution given. The initial condition is simplified by the algebra of the matrices (10.4.28) to

$$\underset{\sim}{G}(0) = \begin{bmatrix} 0 & \tfrac{1}{2}\langle 1 + S_z\rangle_s & -\langle S^+\rangle_s \\ \tfrac{1}{2}\langle 1 - S_z\rangle_s & 0 & \langle S^-\rangle_s \\ \langle S^+\rangle_s & -\langle S^-\rangle_s & 1 \end{bmatrix} \tag{10.5.37}$$

and the stationary values have already been given in Section 10.4.2 The problem of the correlation function and spectrum has been explicitly deduced by *Carmichael* and *Walls* [10.6].

10.6 Generalised *P*-Representations

The idea of representing a density matrix as a linear combination of coherent state density matrices, as in the Glauber-Sudarshan *P*-representation, is particularly useful in quantum systems described by Bose operators a^+ and a, since an interpretation of quantum Master equations in terms of ordinary stochastic processes is only possible this way. This contrasts with the Poisson representation, which merely transforms a birth-death stochastic process into a diffusion process.

Despite the formal similarity to a classical probability distribution, the function $P(\alpha)$ is not a true probability distribution but belongs to a class of quasiprobability distributions. While $P(\alpha)$ exists for thermal light fields (a Gaussian distribution– see Sect. 10.4.1) and coherent laser fields (a δ function distribution), for fields with nonclassical photon statistics, $P(\alpha)$ does not necessarily exist as a well-behaved positive function (although *Klauder* and *Sudarshan* [10.3] have shown that it does exist in terms of extremely singular distributions). Such nonclassical fields have been observed in experiments on atomic fluorescence. Alternative quasiprobability distributions which avoid some of the problems of the P-representation exist. The Wigner function which was the first quasiprobability method, may be obtained from the P-representation by the following integral:

$$W(\alpha) = \frac{2}{\pi} \int P(\beta) \exp(-2|\beta - \alpha|^2) d^2\beta \,. \tag{10.6.1}$$

The Wigner function always exists as a nonsingular function but may assume negative values. The Wigner distribution simplifies averaging *symmetrically-ordered* operator products but is less convenient for averaging the usual *normally-ordered* operator products.

An alternative representation which is always positive is the Q-representation or diagonal matrix elements of the density operator in terms of the coherent states:

$$Q(\alpha, \alpha^*) = \langle \alpha | \rho | \alpha \rangle \,. \tag{10.6.2}$$

Though this representation is positive, it has the disadvantage that not every positive Q function corresponds to a positive semidefinite Hermitian density operator. In addition, evaluating moments is only simple in the Q-representation for *antinormally-ordered* operator products.

Glauber defined an *R-representation* by means of the operator expansion given in Sect. 10.1.2e:

$$\begin{aligned} R(\alpha^*, \beta) &= \langle \alpha | \rho | \beta \rangle \exp(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2) \\ \rho &= \frac{1}{\pi^2} \int |\alpha\rangle R(\alpha^*, \beta) \langle \beta | \exp[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2)] d^2\alpha d^2\beta \,. \end{aligned} \tag{10.6.3}$$

While the representation is analytic in α^*, β (and therefore nonsingular), it is also, by definition, nonpositive and has a normalization that includes a Gaussian weight factor. For this reason, it cannot have a Fokker-Planck equation or any direct interpretation as a quasiprobability. Nevertheless, the existence of this representation does demonstrate that a calculation of normally-ordered observables for any ρ is possible with a nonsingular representation.

10.6.1 Definition of Generalised P-Representation

In order to treat problems in nonlinear quantum optics where nonclassical photon statistics arise, a class of generalized P-representations was introduced by *Drum-*

mond and *Gardiner* [10.9] by expanding in nondiagonal coherent state projection operators. The methods used are very similar to those used for defining various Poisson representations in Sect. 7.7. The generalised P-representations are defined as follows. We set

$$\rho = \int_{\mathscr{D}} \Lambda(\alpha, \beta) P(\alpha, \beta) d\mu(\alpha, \beta) , \tag{10.6.4}$$

where

$$\Lambda(\alpha, \beta) = \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle} , \tag{10.6.5}$$

$d\mu(\alpha, \beta)$ is the integration measure which may be chosen to define different classes of possible representations and $\mathscr{D}$ is the domain of integration. The projection operator $\Lambda(\alpha, \beta)$ is analytic in (α, β).

Useful choices of the integration measure are:

a) Glauber-Sudarshan *P*-Representation

$$d\mu(\alpha, \beta) = \delta^2(\alpha^* - \beta)\, d^2\alpha d^2\beta \tag{10.6.6}$$

This measure corresponds to the Glauber-Sudarshan P-representation.

b) Complex *P*-Representation

$$d\mu(\alpha, \beta) = d\alpha\, d\beta \tag{10.6.7}$$

Here, (α, β) are treated as complex variables which are to be integrated on individual contours C, C'. The existence of this representation under certain circumstances is demonstrated in the next section. In particular, this representation exists for an operator expanded in a finite basis of number states. This is a characteristic situation where nonclassical photon statistics (photon antibunching) may arise, and where the Glauber-Sudarshan P-representation would be singular. This representation is called the complex P-representation since complex values of $P(\alpha,\beta)$ occur. It gives rise to a $P(\alpha, \beta)$ which can be shown to satisfy a FPE obtained by replacing (α, α^*) with (α, β) in the usual Glauber-Sudarshan type of FPE.

Under certain circumstances, exact solutions to Fokker-Planck equations occur which cannot be normalized as Glauber-Sudarshan P-functions. These can be handled with the present representation by choosing C, C' (paths of integration) in the complex phase space of (α, β).

c) Positive *P*-Representation

$$d\mu(\alpha, \beta) = d^2\alpha d^2\beta$$

This representation allows (α, β) to vary independently over the whole complex plane. In the next section we will show that $P(\alpha, \beta)$ always exists for a physical density operator and can always be chosen positive, in which case we call it the positive P-representation. This means that $P(\alpha, \beta)$ has all the properties of a genuine probability. It will also be shown that provided any FPE exists for time development in the Glauber-Sudarshan representation, a corresponding FPE exists with a positive semidefinite diffusion coefficient for the positive *P*-representation. This enables stochastic differential equations, and a correspondence between the quantum Markov process and ordinary diffusion processes to be derived.

In all representations, it is, of course, true that observable moments are given by

$$\langle (a^+)^m a^n \rangle = \int_{\mathcal{D}} d\mu(\alpha, \beta)\beta^m \alpha^n P(\alpha, \beta) . \tag{10.6.8}$$

10.6.2 Existence Theorems

We will show in this section that the generalised *P*-representations have quite strong existence properties. We do this with a number of theorems. NB: for brevity, we shall use the notation

$$\underline{\alpha} = (\alpha, \beta) .$$

Theorem 1: A complex *P*-representation exists for an operator with an expansion in a finite number of number states.

Proof: Let

$$\rho = \sum_{n,m}\sum C_{nm}(a^+)^m |0\rangle\langle 0| a^n . \tag{10.6.9}$$

Then, by Cauchy's theorem,

$$\rho = \oint\oint_{C,C'} \Lambda(\underline{\alpha})P(\underline{\alpha})d\mu(\underline{\alpha}) \tag{10.6.10}$$

with

$$P(\underline{\alpha}) = (-1/4\pi^2)e^{\alpha\beta} \sum_{n,m}\sum C_{nm}\, n!m!\alpha^{-m-1}\beta^{-n-1} \tag{10.6.11}$$

where C, C' are integration paths enclosing the origin.

Theorem 2: A complex *P*-representation exists for any operator with an expansion on a bounded range of coherent states, i.e., for

$$\rho = \int\int_{D,D'} \Lambda(\alpha, \beta)C(\alpha, \beta)d^2\alpha d^2\beta , \tag{10.6.12}$$

where D, D' are bounded in each complex plane.

Proof: Application of Cauchy's theorem shows that if

$$P(\underline{\alpha}) = -\frac{1}{4\pi^2}\int\int_{D,D'} C(\alpha', \beta')/[(\alpha - \alpha')(\beta - \beta')]d^2\alpha' d^2\beta' , \tag{10.6.13}$$

then

$$\rho = \iint_{C,C'} \Lambda(\underset{\sim}{\alpha})P(\underset{\sim}{\alpha})d\alpha\, d\beta\,, \tag{10.6.14}$$

where C, C' enclose D, D' respectively. Hence, the complex P-representation exists in this case relative to any bounded expansion in coherent state projection operators.

Theorem 3: A positive P-representation exists for any quantum density operator ρ, with

$$P(\underset{\sim}{\alpha}) = (1/4\pi^2)\exp(-|\alpha - \beta^*|^2/4)\langle\tfrac{1}{2}(\alpha + \beta^*)|\rho|\tfrac{1}{2}(\alpha + \beta^*)\rangle\,. \tag{10.6.15}$$

Proof: $P(\underset{\sim}{\alpha})$ is positive, since ρ is a density operator, and it is composed of a diagonal matrix element multiplied by a positive function. In order to show that this represents a quantum density operator in the general case, the characteristic function

$$\chi(\lambda) \equiv \mathrm{Tr}\{\rho\, e^{\lambda a^+}\, e^{-\lambda^* a}\} \tag{10.6.16}$$

is used. This has been shown in Sect. 10.2.5 to define the density operator uniquely. In terms of the R-representation for $\hat{\rho}$, the characteristic function is

$$\chi(\lambda) = \frac{1}{\pi}\int R(\alpha^*, \lambda + \alpha)\exp(-\lambda^*\alpha - |\alpha|^2)d^2\alpha\,. \tag{10.6.17}$$

We now substitute the R-representation for ρ into (10.6.15) which defines $P(\underset{\sim}{\alpha})$ in terms of the diagonal matrix elements of ρ. We then define ρ_P to be given by the positive P-representation form (10.6.4) with $P(\underset{\sim}{\alpha})$ as given by the previous process, calculate the corresponding characteristic function $\chi_P(\lambda)$ using (10.6.16) and show that this is the same as the original characteristic function for ρ. Thus:

$$\begin{aligned}\chi_P(\chi) &= \iint P(\underset{\sim}{\alpha})\exp(\lambda\beta - \lambda^*\alpha)d^2\alpha d^2\beta\\ &= \frac{1}{4\pi^4}\iiiint R(\alpha'^*, \beta')\exp[\lambda\beta - \lambda^*\alpha - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 - |\alpha'|^2 - |\beta'|^2\\ &\quad + \tfrac{1}{2}\beta'^*(\alpha + \beta^*) + \tfrac{1}{2}\alpha'^*(\alpha^* + \beta)]d^2\alpha d^2\beta d^2\alpha' d^2\beta'\,.\end{aligned}$$

We now make a variable change by defining

$$\begin{aligned}&\gamma = (\alpha + \beta^*)/2 \qquad \delta = (\alpha - \beta^*)/2\\ &\alpha = (\gamma + \delta) \qquad\quad \beta^* = (\gamma - \delta)\\ &d^2\alpha d^2\beta = 4d^2\gamma d^2\delta\,.\end{aligned} \tag{10.6.18}$$

Noting that R is an analytic function, the following identify is useful:

$$R(\alpha^*, \gamma) = \frac{1}{\pi}\int R(\alpha^*, \beta)\exp(\gamma\beta^* - |\beta|^2)d^2\beta\,. \tag{10.6.19}$$

Hence, the above expression for the characteristic function can be simplified to give

$$\chi_P(\lambda) = \frac{1}{\pi^3} \int\int\int R(\alpha'^*, \gamma) \exp\,[\lambda(\gamma - \delta)^* - \lambda^*(\gamma + \delta)$$

$$- |\gamma|^2 - |\delta|^2 - |\alpha'|^2 + \alpha'\gamma^*] d^2\gamma d^2\delta d^2\alpha' \tag{10.6.20}$$

$$= \frac{1}{\pi^2} \int\int R(\alpha'^*, \gamma) \exp\,(|\lambda|^2 + \lambda\gamma^* - \lambda^*\gamma - |\gamma|^2$$

$$- |\alpha'|^2 + \alpha'\gamma^*) d^2\gamma d^2\alpha' \tag{10.6.21}$$

$$= \frac{1}{\pi} \int R(\alpha^*, \lambda + \alpha) \exp\,(-\lambda^*\alpha - |\alpha|^2) d^2\alpha\,. \tag{10.6.22}$$

Hence,

$$\chi_P(\lambda) = \mathrm{Tr}\,\{\rho e^{\lambda a^+} e^{-\lambda^* a}\} = \chi(\lambda)\,. \tag{10.6.23}$$

The last step follows from the identity with the characteristic function defined relative to the *R*-representation in (10.6.17), Thus, we deduce that $\rho_P = \rho$.

10.6.3 Relation to Poisson Representation

Given a probability distribution $q(x)$ over the integers, we can always define a corresponding positive density matrix by

$$\rho = \sum_n |n\rangle \langle n| q(n) \tag{10.6.24}$$

and a *P*-representation for ρ gives the corresponding Poisson representation for $P(n)$; thus,

$$q(x) = \langle x|\rho|x\rangle = \langle x| \int d\mu(\alpha, \beta) P(\alpha,\beta) \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle} |x\rangle \tag{10.6.25}$$

$$= \int d\mu(\alpha, \beta) \frac{e^{-\alpha\beta}(\alpha\beta)^x}{x!} P(\alpha, \beta)\,. \tag{10.6.26}$$

Hence, one can write

$$q(x) = \int d\mu(\alpha_1) f(\alpha_1) \frac{e^{-\alpha_1}\alpha_1{}^x}{x!} \tag{10.6.27}$$

with

$$f(\alpha_1) d\mu(\alpha_1) = \int d\mu(\alpha, \beta) \delta_\mu(\alpha\beta - \alpha_1) P(\alpha, \beta) \tag{10.6.28}$$

and $\delta_\mu(\alpha_1 - \alpha_2)$ is a Dirac delta function defined with respect to the measure $\mu(\alpha_1)$, i.e.,

$$\int d\mu(\alpha_1) \delta_\mu(\alpha_1 - \alpha_2) \phi(\alpha_1) = \phi(\alpha_2)\,. \tag{10.6.29}$$

Thus, we can deduce from Theorem 3 that a positive Poisson representation always exists, as asserted in Sect. 7.7.4. The first theorem can also be adapted to show that a complex P-representation always exists if $q(x) = 0$ for $x > N$, for some finite N. However, a more general result has already been proved in Sect. 7.7.3.

10.6.4 Operator Identities

From the definitions (10.6.5) of the nondiagonal coherent state projection operators, the following identities can be obtained. Again, $\underset{\sim}{\alpha}$ is used to denote (α, β):

$$\left.\begin{aligned} &a\Lambda(\underset{\sim}{\alpha}) = \alpha\Lambda(\underset{\sim}{\alpha}) \\ &a^{+}\Lambda(\underset{\sim}{\alpha}) = (\beta + \partial/\partial\alpha)\Lambda(\underset{\sim}{\alpha}) \\ &\Lambda(\underset{\sim}{\alpha})a^{+} = \beta\Lambda(\underset{\sim}{\alpha}) \\ &\Lambda(\underset{\sim}{\alpha})a = (\partial/\partial\beta + \alpha)\Lambda(\underset{\sim}{\alpha})\,. \end{aligned}\right\} \tag{10.6.30}$$

By substituting the above identities into (10.6.4) defining the generalised P-representation and using partial integration (provided the boundary terms vanish), these identities can be used to generate operations on the P-function depending on the representation.

a) Glauber-Sudarshan *P*-Representation
The same results as (10.2.26).

b) Complex *P*-Representation

$$\left.\begin{aligned} &a\rho \leftrightarrow \alpha P(\underset{\sim}{\alpha}) \\ &a^{+}\rho \leftrightarrow (\beta - \partial/\partial\alpha)P(\underset{\sim}{\alpha}) \\ &\rho a^{+} \leftrightarrow \beta P(\underset{\sim}{\alpha}) \\ &\rho a \leftrightarrow (\alpha - \partial/\partial\beta)P(\underset{\sim}{\alpha}) \end{aligned}\right\} \tag{10.6.31}$$

c) Positive *P*-Representation
We now use the analyticity of $\Lambda(\alpha, \beta)$ and note that if

$$\begin{aligned} \alpha &= \alpha_x + \mathrm{i}\alpha_y \\ \beta &= \beta_x + \mathrm{i}\beta_y\,, \end{aligned} \tag{10.6.32}$$

then

$$(\partial/\partial\alpha)\Lambda(\underset{\sim}{\alpha}) = (\partial/\partial\alpha_x)\Lambda(\underset{\sim}{\alpha}) = (-\mathrm{i}\partial/\partial\alpha_y)\Lambda(\underset{\sim}{\alpha}) \tag{10.6.33}$$

and

$$(\partial/\partial\beta)\Lambda(\underset{\sim}{\alpha}) = (\partial/\partial\beta_x)\Lambda(\underset{\sim}{\alpha}) = (-\mathrm{i}\partial/\partial\beta_y)\Lambda(\underset{\sim}{\alpha}) \tag{10.6.34}$$

so that as well as all of (10.6.31) being true in this case, we also have

$$\begin{aligned} a^+\hat{\rho} &\leftrightarrow (\beta - \partial/\partial\alpha_x)P(\underset{\sim}{\alpha}) \leftrightarrow (\beta + \mathrm{i}\partial/\partial\alpha_y)P(\underset{\sim}{\alpha}) \\ \rho a &\leftrightarrow (\alpha - \partial/\partial\beta_x)P(\underset{\sim}{\alpha}) \leftrightarrow (\alpha + \mathrm{i}\partial/\partial\beta_y)P(\underset{\sim}{\alpha})\,. \end{aligned} \tag{10.6.35}$$

All these correspondences can now be used to derive Fokker-Planck equations when appropriate.

10.7 Application of Generalised *P*-Representations to Time-Development Equations

We firstly want to show that the Glauber-Sudarshan *P*-representation will not always yield an acceptable Fokker-Planck equation, and that realistic situations arise in which this is the case.

Consider a coherently driven single mode interferometer with a nonlinear absorber, for which we set [10.9]

$$H_1 = \sum_i h\omega_i(\Gamma_i^+\Gamma_i + \tfrac{1}{2}) \tag{10.7.1}$$

$$H_2 = \sum_i [(a^+)^2 g_i \Gamma_i + a^2 g_i{}^* \Gamma_i^+] \tag{10.7.2}$$

$$H_3 = h\omega(a^+a + \tfrac{1}{2}) + \mathrm{i}h(\varepsilon \mathrm{e}^{-\mathrm{i}\omega t}a^+ - \varepsilon^*\,\mathrm{e}^{\mathrm{i}\omega t}a)\,. \tag{10.7.3}$$

This is much the same as the example treated in Sect. 10.4.1. In an interaction picture, we get the Master equation (assuming a zero temperature heat bath) and using (10.3.38),

$$\boxed{\frac{\partial\rho}{\partial t} = [\varepsilon a^+ - \varepsilon^* a, \rho] + \tfrac{1}{2}K[2a^2\rho(a^+)^2 - (a^+)^2a^2\rho - \rho(a^+)^2a^2]\,.} \tag{10.7.4}$$

Using the ordinary Glauber-Sudarshan operator correspondences, we would obtain

$$\frac{\partial}{\partial t}P(\alpha,\alpha^*) = \left[-\frac{\partial}{\partial\alpha}(\varepsilon - K\alpha^2\alpha^*) - \tfrac{1}{2}\frac{\partial^2}{\partial\alpha^2}(K\alpha^2) + \text{comp. conj.}\right]P(\alpha,\alpha^*). \tag{10.7.5}$$

In terms of real variables,

$$\begin{aligned} x &= (\alpha + \alpha^*)/\sqrt{2} \\ y &= (\alpha - \alpha^*)/i\sqrt{2}\,. \end{aligned}$$

The diffusion matrix is

$$\begin{bmatrix} -K/2 & -K \\ -K & K/2 \end{bmatrix} \tag{10.7.6}$$

which is not positive semidefinite. Hence, a time development equation of the form

(10.7.4) will develop singularities. We are lead, therefore, to alternative equations in the various P-representations.

Naively following the rules which would apply if there was a positive semidefinite diffusion matrix leads to the stochastic differential equations

$$\frac{d}{dt}\begin{bmatrix}\alpha \\ \alpha^*\end{bmatrix} = \begin{bmatrix}\varepsilon - K\alpha^2\alpha^* \\ \varepsilon - K(\alpha^*)^2\alpha\end{bmatrix} + \mathrm{i}\sqrt{K}\begin{bmatrix}\alpha\xi_1(t) \\ \alpha^*\xi_2(t)\end{bmatrix}, \tag{10.7.7}$$

where $\xi_1(t)$ and $\xi_2(t)$ are independent white noises. However, a paradox arises. Because ξ_1 and ξ_2 are independent, α and α^* do not remain complex conjugate. We are lead to a similar situation to that of the Poisson representation, where negative diffusion matrices also turn up.

We will show that (10.7.7) is, in fact, correct, provided we make the replacement $\alpha^* \to \beta$ and they are regarded as variables of the positive P-representation.

10.7.1 Complex P-Representation

Here the procedure yields a very similar equation to that for the Glauber-Sudarshan case. We assume that, by appropriate re-ordering of the differential operators, We can reduce the quantum mechanical master equation to the form [where $(\alpha, \beta) = \underset{\sim}{\alpha} \equiv (\alpha^{(1)}, \alpha^{(2)})$; $\mu = 1, 2$]:

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \iint_{C,C'} \Lambda(\underset{\sim}{\alpha})\frac{\partial P(\underset{\sim}{\alpha})}{\partial t}\, d\alpha\, d\beta \\ &= \iint_{C,C'} \left\{\left[A^\mu(\underset{\sim}{\alpha})\frac{\partial}{\partial\alpha^\mu} + \tfrac{1}{2}D^{\mu\nu}(\underset{\sim}{\alpha})\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\alpha^\nu}\right]\Lambda(\underset{\sim}{\alpha})\right\}P(\underset{\sim}{\alpha})d\alpha\, d\beta\,. \end{aligned}\tag{10.7.8}$$

We now integrate by parts and, if we can neglect boundary terms which may be made possible by an appropriate choice of contours C, C', at least one solution is obtained by equating the coefficients of $\Lambda(\underset{\sim}{\alpha})$:

$$\frac{\partial P(\underset{\sim}{\alpha})}{\partial t} = \left[-\frac{\partial}{\partial\alpha^\mu}A^\mu(\underset{\sim}{\alpha}) + \tfrac{1}{2}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\alpha^\nu}D^{\mu\nu}(\underset{\sim}{\alpha})\right]P(\underset{\sim}{\alpha})\,. \tag{10.7.9}$$

This equation is sufficient to imply (10.7.8), but is not a unique equation becuase the $\Lambda(\underset{\sim}{\alpha})$ are not linearly independent. It should be noted that for this complex P-representation, $A^\mu(\underset{\sim}{\alpha})$ and $D^{\mu\nu}(\underset{\sim}{\alpha})$ are always analytic in $\underset{\sim}{\alpha}$; hence, if $P(\underset{\sim}{\alpha})$ is initially analytic, (10.7.9) preserves this analyticity as time develops.

10.7.2 Positive P-Representation

We assume that the same equation (10.7.8) is being considered but with a positive P-representation. The symmetric matrix can always be factorised into the form

$$\underset{\sim}{D}(\underset{\sim}{\alpha}) = \underset{\sim}{B}(\underset{\sim}{\alpha})\underset{\sim}{B}^{\mathrm{T}}(\underset{\sim}{\alpha})\,. \tag{10.7.10}$$

We now write

$$\boldsymbol{A}(\underset{\sim}{\alpha}) = \boldsymbol{A}_x(\underset{\sim}{\alpha}) + \mathrm{i}\boldsymbol{A}_y(\underset{\sim}{\alpha}) \tag{10.7.11}$$

$$\underset{\sim}{\boldsymbol{B}}(\underset{\sim}{\alpha}) = \underset{\sim}{\boldsymbol{B}}_x(\underset{\sim}{\alpha}) + \mathrm{i}\underset{\sim}{\boldsymbol{B}}_y(\underset{\sim}{\alpha})\,, \tag{10.7.12}$$

where $\boldsymbol{A}_x$, $\boldsymbol{A}_y$, $\underset{\sim}{\boldsymbol{B}}_x$, $\underset{\sim}{\boldsymbol{B}}_y$ are real. We then find that the Master equation yields

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \iint d^2\alpha d^2\beta \Lambda(\underset{\sim}{\alpha})\,(\partial P(\underset{\sim}{\alpha})/\partial t)\\ &= \iint P(\underset{\sim}{\alpha})[A_x^\mu(\underset{\sim}{\alpha})\partial_\mu^x + A_y^\mu(\underset{\sim}{\alpha})\partial_\mu^y + \tfrac{1}{2}(B_x^{\mu\sigma}B_x^{\nu\sigma}\partial_\mu^x\partial_\nu^x\\ &\quad + B_y^{\mu\sigma}B_y^{\nu\sigma}\partial_\mu^y\partial_\nu^y + 2B_x^{\mu\sigma}B_y^{\nu\sigma}\partial_\mu^x\partial_\nu^y)]\Lambda(\underset{\sim}{\alpha})d^2\alpha d^2\beta\,.\end{aligned} \tag{10.7.13}$$

Here we have written, for notational simplicity, $\partial/\partial\alpha_x^\mu = \partial_\mu^x$, etc, and have used the analyticity of $\Lambda(\underset{\sim}{\alpha})$ to make either of the replacements

$$\partial/\partial\alpha^\mu \leftrightarrow \partial_\mu^x \leftrightarrow -\mathrm{i}\partial_\mu^y \tag{10.7.14}$$

in such a way as to yield (10.7.13). Now, provided partial integration is permissible, we deduce the FPE

$$\begin{aligned}\partial P(\underset{\sim}{\alpha})/\partial t = \{&-\partial_\mu^x A_x^\mu(\underset{\sim}{\alpha}) - \partial_\mu^y A_y^\mu(\underset{\sim}{\alpha}) + \tfrac{1}{2}[\partial_\mu^x\partial_\nu^x B_x^{\mu\sigma}(\underset{\sim}{\alpha})B_x^{\nu\sigma}(\underset{\sim}{\alpha})\\ &+ 2\partial_\mu^x\partial_\nu^y B_x^{\mu\sigma}(\underset{\sim}{\alpha})B_y^{\nu\sigma}(\underset{\sim}{\alpha}) + \partial_\mu^y\partial_\nu^y B_y^{\mu\sigma}(\underset{\sim}{\alpha})B_y^{\nu\sigma}(\underset{\sim}{\alpha})]\}\,P(\underset{\sim}{\alpha})\,.\end{aligned} \tag{10.7.15}$$

Again, this is not a unique time-development equation but (10.7.13) is a consequence of (10.7.15)

However, the FPE (10.7.15) now possesses a positive semidefinite diffusion matrix in a four-dimensional space whose vectors are

$$(\alpha_x^{(1)}, \alpha_x^{(2)}, \alpha_y^{(1)}, \alpha_y^{(2)}) \equiv (\alpha_x, \beta_x, \alpha_y, \beta_y)\,. \tag{10.7.16}$$

We find the drift vector is

$$\mathscr{A}(\underset{\sim}{\alpha}) \equiv (A_x^{(1)}(\underset{\sim}{\alpha}), A_x^{(2)}(\underset{\sim}{\alpha}), A_y^{(1)}(\underset{\sim}{\alpha}), A_y^{(2)}(\underset{\sim}{\alpha})) \tag{10.7.17}$$

and the diffusion matrix is

$$\mathscr{D}(\underset{\sim}{\alpha}) = \begin{bmatrix}\underset{\sim}{\boldsymbol{B}}_x\underset{\sim}{\boldsymbol{B}}_x^{\mathrm{T}}, & \underset{\sim}{\boldsymbol{B}}_x\underset{\sim}{\boldsymbol{B}}_y^{\mathrm{T}}\\ \underset{\sim}{\boldsymbol{B}}_y\underset{\sim}{\boldsymbol{B}}_x^{\mathrm{T}}, & \underset{\sim}{\boldsymbol{B}}_y\underset{\sim}{\boldsymbol{B}}_y^{\mathrm{T}}\end{bmatrix}(\underset{\sim}{\alpha}) \equiv \mathscr{B}(\underset{\sim}{\alpha})\mathscr{B}^{\mathrm{T}}(\underset{\sim}{\alpha})\,, \tag{10.7.18}$$

where

$$\mathscr{B}(\underset{\sim}{\alpha}) = \begin{bmatrix}\underset{\sim}{\boldsymbol{B}}_x, & 0\\ \underset{\sim}{\boldsymbol{B}}_y, & 0\end{bmatrix}(\underset{\sim}{\alpha}) \tag{10.7.19}$$

and $\mathscr{D}$ is thus explicitly positive semidefinite (and not positive definite). The corresponding Ito stochastic differential equations can be written as

$$\frac{d}{dt}\begin{bmatrix}\underline{\alpha}_x\\ \underline{\alpha}_y\end{bmatrix}=\begin{bmatrix}A_x(\underline{\alpha})\\ A_y(\underline{\alpha})\end{bmatrix}+\begin{bmatrix}\underline{B}_x(\underline{\alpha})\xi(t)\\ \underline{B}_y(\underline{\alpha})\xi(t)\end{bmatrix}, \tag{10.7.20}$$

or recombining real and imaginary parts,

$$d\underline{\alpha}/dt = A(\underline{\alpha}) + \underline{B}(\underline{\alpha})\xi(t)\,. \tag{10.7.21}$$

Apart from the substitution $\alpha^* \to \beta$, (10.7.21) is just the stochastic differential equation which would be obtained by using the Glauber-Sudarshan representation and naively converting the Fokker-Planck equation with a non-positive-definite diffusion matrix into an Ito stochastic differential equation.

In our derivation, the two formal variables (α, α^*) have been replaced by variables in the complex plane (α, β) that are allowed to fluctuate independently. The positive P-representation as defined here thus appears as a mathematical justification of this procedure. The procedure used closely parallels that of Sect. 7.7.4. on the positive Poisson representation.

10.7.3 Example

We consider the example of Sect. 10.7. Using the appropriate operator correspondence, the complex P-representation FPE is

$$\begin{aligned}\partial_t P(\alpha,\beta) = \Big[&-\frac{\partial}{\partial\alpha}(\varepsilon - K\alpha^2\beta) - \frac{1}{2}\frac{\partial^2}{\partial\alpha^2}(K\alpha^2) - \frac{\partial}{\partial\beta}(\varepsilon - K\alpha\beta^2)\\ &-\frac{1}{2}\frac{\partial^2}{\partial\beta^2}(K\beta^2)\Big]P(\alpha,\beta)\,.\end{aligned} \tag{10.7.22}$$

Rather miraculously, we see that this FPE satisfies potential conditions of Sect. 5.3.3. For, in that notation,

$$A=\begin{bmatrix}\varepsilon - K\alpha^2\beta\\ \varepsilon - K\beta^2\alpha\end{bmatrix},\qquad \underline{B}=\begin{bmatrix}-K\alpha^2 & 0\\ 0 & -K\beta^2\end{bmatrix} \tag{10.7.23}$$

so, using (5.3.22,23),

$$\begin{aligned}&Z=-\frac{2}{K}\begin{bmatrix}\varepsilon/\alpha^2 - K\beta + K/\alpha\\ \varepsilon/\beta^2 - K\alpha + K/\beta\end{bmatrix},\\ &\frac{\partial Z_\alpha}{\partial\beta}=\frac{\partial Z_\beta}{\partial\alpha}=2\end{aligned} \tag{10.7.24}$$

and

$$\begin{aligned}\phi(\alpha,\beta) &= -\int(Z_\alpha d\alpha + Z_\beta d\beta)\\ &= -\frac{2\varepsilon}{K}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) + 2\log(\alpha\beta) - 2\alpha\beta\,,\end{aligned} \tag{10.7.25}$$

so that

$$\boxed{P_s(\alpha, \beta) = (\alpha\beta)^{-2} \exp\left[2\alpha\beta + \frac{2\varepsilon}{K}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right].} \tag{10.7.26}$$

The only acceptable contours for this stationary distribution are C, C' which are independent contours in the α and β planes which encircle the essential singularities at $\alpha = 0$ and $\beta = 0$.

A potential solution of this kind is extremely useful and could not be obtained with the Glauber-Sudarshan P-representation. The moments can be obtained from

$$\iint d\alpha\, d\beta\, \beta^m \alpha^n (\alpha\beta)^{-2} \exp\left[2\alpha\beta + \frac{2\varepsilon}{K}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right] \tag{10.7.27}$$

and we can expand $\exp(2\alpha\beta)$ in power series and contour integrate term by term, to obtain

$$\boxed{\langle (a^+)^m a^n \rangle = \sum_{r=0}^{\infty} \frac{\left(\frac{2\varepsilon}{K}\right)^{n+m-2r-2} 2^r}{r!(n-r-1)!\,(m-r-1)!}} \tag{10.7.28}$$

which is an easily computed series.

Using the positive P-representation, we obtain the stochastic differential equation

$$\begin{bmatrix} d\alpha \\ d\beta \end{bmatrix} = \begin{bmatrix} \varepsilon - K\alpha^2\beta \\ \varepsilon - K\alpha\beta^2 \end{bmatrix} dt + \mathrm{i}\sqrt{K} \begin{bmatrix} \alpha\, dW_1(t) \\ \beta\, dW_2(t) \end{bmatrix}. \tag{10.7.29}$$

It should be noted that this equation does not contain any very obvious small noise parameter. However, a large driving field limit can be obtained by setting

$$\begin{aligned} K &= \tilde{K}/\varepsilon^2 \\ \alpha &= \tilde{\alpha}\varepsilon \\ \beta &= \tilde{\beta}\varepsilon \end{aligned} \tag{10.7.30}$$

so that

$$\begin{bmatrix} d\tilde{\alpha} \\ d\tilde{\beta} \end{bmatrix} = \begin{bmatrix} 1 - \tilde{K}\tilde{\alpha}^2\tilde{\beta} \\ 1 - \tilde{K}\tilde{\alpha}\tilde{\beta}^2 \end{bmatrix} dt + \mathrm{i}\frac{\sqrt{\tilde{K}}}{\varepsilon} \begin{bmatrix} \tilde{\alpha}\, dW_1(t) \\ \tilde{\beta}\, dW_2(t) \end{bmatrix}. \tag{10.7.31}$$

A small noise linearisation process can be carried out in this limit of large driving field and small nonlinearity which is, in fact, a situation of practical utility.